A representation of finite, positive, commutative tomonoids

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Abstract

We propose a representation of totally ordered monoids (or tomonoids, for short) that are commutative, positive, and finite. We identify these structures with compatible and positive total preorders on $\mathbb{N}^n$. To describe the latter, we utilise so-called direction f-cones, making use of the special form of the associated congruences on $\mathbb{N}^n$.

1 Introduction

Totally ordered monoids, or tomonoids as we say shortly [EKMMW], naturally occur in the context of fuzzy logic. In contrast to classical logic, which is based on the two truth values "false" and "true", fuzzy logic allows propositions to be assigned also intermediary degrees; see, e.g., [Háj]. The conjunction is moreover interpreted by an operation that acts on the chain of generalised truth values. The actual interpretation varies from logic to logic and is often not restricted to a single choice. A common requirement is associativity as well as the compatibility with the total order. We are in this case led to totally ordered semigroups [Cli, Gab]. A further assumption may be that the top element of the chain represents "true" and thus should be an identity, in which case the relevant structures are negative tomonoids. If, in addition, for the conjunction the order of propositions does not matter, the tomonoids of interest are also commutative [EKMMW].

On the basis of such motivation, we investigate in this paper negative, commutative tomonoids. We use the additive notation and work with the dual order; hence we actually call the structures in question positive commutative (p.c.) tomonoids. We furthermore restrict to the finite case. A further requirement common in fuzzy logic is then automatically fulfilled: residuation [Háj, GJKO]. Adding the residual implication, which is uniquely determined, we obtain totally ordered MTL-algebras [EsGo]. Hence our work could also be understood as a contribution to a better understanding of finite MTL-chains.

In recent years, residuated chains have been studied under various assumptions from various points of view. For instance, MTL-chains with the weak cancellation property are the topic of [MNH] as well as [Hor1]. Idempotent residuated chains are studied in [ChZh]. We may moreover mention our paper [Vet1], devoted to MTL-algebras that are based on the real unit interval. The content of certain works comes even very close to what we are concerned with here, although the chosen approaches are much different. The paper [Hor2], for instance, deals with finite MTL-chains and their relationship to Abelian totally ordered groups. Finally, a step-wise construction of finite p.c. tomonoids is proposed in [PeVe].

The present work is to be seen in the following context. Disregarding the total order, we deal with finite, commutative monoids. A commutative monoid can be described by a monoid congruence on $\mathbb{N}^n$, where n is the cardinality of a generating subset. Congruences of commutative semigroups in general and of free commutative monoids in particular have been described, e.g., in the well-known paper by Eilenberg and Schützenberger [EiSc]. The representation of monoids by congruences of $\mathbb{N}^n$ is the starting point of the present study. We will see that, in our special context, the description of the congruences on $\mathbb{N}^n$ can actually be simplified.

Our task is to describe a quotient of $\mathbb{N}^n$ together with a total order that is compatible with the addition. To this end, we make use of an idea introduced in our previous paper [Vet2]. If L is a tomonoid and $\varphi\colon \mathbb{N}^n \to L$ is a surjective homomorphism of monoids, we may pull back the total order $\leqslant$ on L to a binary relation $\preccurlyeq$ on $\mathbb{N}^n$: we put $a \preccurlyeq b$ if $\varphi(a) \leqslant \varphi(b)$. Then $\preccurlyeq$ is a preorder that encodes both the monoid congruence and the total order. In fact, we obtain the congruence by identifying any pair $a, b \in \mathbb{N}^n$ such that $a \preccurlyeq b$ and $b \preccurlyeq a$. A preorder $\preccurlyeq$ on $\mathbb{N}^n$ represents a finitely generated, positive, commutative tomonoid if and only if $\preccurlyeq$ is positive, compatible, and total. Such preorders, which we call *monomial*, are consequently the central topic of the present paper.

In order to describe monomial preorders, the following considerations turn out to be useful. If $\preccurlyeq$ happens to be a total order on $\mathbb{N}^n$, the set of differences $b - a$ such that $a \preccurlyeq b$ is the positive cone of a totally ordered group based on $\mathbb{Z}^n$ and the positive cone determines the total order uniquely. In the general case, this set seems to be of limited use. We may, however, slightly modify its definition and consider the set of all $z \in \mathbb{Z}^n$ such that $a \preccurlyeq b$ whenever $b - a = z$. We call the latter set a *direction cone*. Direction cones can be characterised in a way inspired by the case of totally ordered Abelian groups and any finitely generated p.c. tomonoid is a quotient of a tomonoid associated with a direction cone [Vet2].

Direction cones are infinite and thus not an elegant tool to deal with the finite case. The present paper aims at overcoming this drawback. To this end, the aforementioned structure of the congruence classes comes into play. Given a monomial preorder $\preccurlyeq$ on $\mathbb{N}^n$, we define a certain finite subset of $\mathbb{N}^n$, called the *support* of $\preccurlyeq$, which comprises all finite classes and has a non-empty intersection also with each infinite class. We show that the preorder $\preccurlyeq$ is uniquely determined by the restriction to its support.

Restricting the considerations to the support, we are led to a finitary analogue of direction cones, called *direction f-cones*. A direction f-cone represents a finite monomial preorder and hence a finite p.c. tomonoid, and any finite p.c. tomonoid is a quotient of a tomonoid obtained in this way. To characterise direction f-cones is not straightforward, however. We offer two perspectives on the problem. We adopt, on the one hand, a "global" viewpoint, considering the direction f-cone as a whole. We proceed, on the other hand, according to a "local" viewpoint, describing the direction f-cone relative to the direction f-cone associated with a tomonoid whose number of Archimedean classes is by one smaller. The former possibility is easier to comprehend. The latter, however, which describes the relevant structure in a step-wise fashion, offers a deeper analysis of the monomial preorders in question.

The paper is structured as follows. We recall in Section 2 the basic fact on which the paper is built, namely, the correspondence of finitely generated p.c. tomonoids and monomial preorders. We moreover introduce to direction cones as a tool of describing monomial preorders. In Section 3, we turn to the finite case. We provide a description of the associated congruences on $\mathbb{N}^n$ and propose a finitary analogy of direction cones, called direction f-cones. In Section 4, we compile the latter's basic properties and we derive a first axiomatic construction of finite monomial preorders. The remaining two sections are devoted to a more detailed analysis of direction f-cones. In Section 5, we describe a direction f-cone relative to the direction f-cone that arises from "collapsing" the smallest Archimedean class. The converse procedure is specified in Section 6 and leads to a second axiomatic construction of finite monomial preorders. Some concluding remarks are contained in Section 7.

2 Monomial preorders and direction cones

We investigate in this paper structures of the following type.

Definition 2.1. A structure $(L; \leqslant, +, 0)$ is a *totally ordered monoid*, or *tomonoid* for short, if (i) $(L; +, 0)$ is a monoid, (ii) $(L; \leqslant)$ is a chain, and (iii) $\leqslant$ is compatible with $+$, that is, $a \leqslant b$ and $c \leqslant d$ imply $a + c \leqslant b + d$.

A tomonoid $(L; \leqslant, +, 0)$ is called *commutative* if so is $+$. Moreover, L is called *positive* if 0 is the bottom element.

The notion of a "tomonoid", in analogy to the better known "pomonoid", is taken from [EKMMW]. We note that, in contrast to [EKMMW], we do not define tomonoids to be commutative. We will rather make the assumption of commutativity explicit and to keep the notation short we will abbreviate "positive, commutative" by "p.c.".

By a *subtomonoid* of a tomonoid L, we mean a submonoid of L endowed with the total order inherited from L. A downwards closed subtomonoid is called an *ideal*.

Congruences of tomonoids are defined in the expected way. Here, a subset C of a poset is called *convex* if $a \leqslant b \leqslant c$ and $a, c \in C$ imply $b \in C$.

Definition 2.2. Let $(L; \leqslant, +, 0)$ be a tomonoid. A *tomonoid congruence* is a monoid congruence $\sim$ on L such that all $\sim$-classes are convex. We endow the quotient $\langle L \rangle_\sim$ with the partial order $\leqslant$, where $\langle a \rangle_\sim \leqslant \langle b \rangle_\sim$ if $a \sim b$ or $a < b$, with the induced operation $+$, and with the constant $\langle 0 \rangle_\sim$.

Lemma 2.3. *Let $\sim$ be a tomonoid congruence on the tomonoid $(L; \leqslant, +, 0)$. Then $(\langle L \rangle_\sim; \leqslant, +, \langle 0 \rangle_\sim)$ is again a tomonoid. Moreover, if L is positive, so is $\langle L \rangle_\sim$, and if L is commutative, so is $\langle L \rangle_\sim$.*

Congruences of tomonoids are not easy to classify. However, there are certain special kinds of congruences that have turned out to be particularly useful. For instance, Rees quotients of finite positive tomonoids were explored in [PeVe]. In the present context, we need the following construction [BlTs, NEG].

Proposition 2.4. *Let F be an ideal of the positive tomonoid L. We define, for any $a, b \in L$,*

$$a \sim_F b \quad \text{if there is an } f \in F \text{ such that } b \leqslant a + f \text{ and } a \leqslant b + f.$$

Then $\sim_F$ is a tomonoid congruence.

Given the ideal F of a positive tomonoid L, we call the quotient $\langle L \rangle_{\sim_F}$ simply the quotient of L by F. In the finite case, quotients by ideals correspond to certain subalgebras. Recall that an element e of a monoid is called *idempotent* if $e + e = e$.

Lemma 2.5. *Let F be an ideal of the positive tomonoid L and assume that F possesses a greatest element e. Then e is idempotent. Moreover, $L_e = \{a + e \colon a \in L\}$ is a subalgebra of L and the mapping*

$$\langle L \rangle_{\sim_F} \to L_e, \quad \langle a \rangle_{\sim_F} \mapsto a + e$$

is an isomorphism of tomonoids.

The tomonoid consisting of the 0 alone is called *trivial*. We say that the $n \geqslant 1$ non-zero elements $g_1, \ldots, g_n$ of a non-trivial tomonoid L *generate* L if they generate L as a monoid.

Following [Vet2], we will represent tomonoids on the basis of congruences of free commutative monoids. The free commutative monoid over $n \geqslant 1$ elements will be identified with $\mathbb{N}^n$. The addition is component-wise and the identity is $\bar{0} = (0, ..., 0)$, the n-tuple consisting solely of 0's. $\mathbb{N}^n$ is obviously generated by the unit vectors $u_1, \ldots, u_n$. Here, $u_i = (0, ..., 0, 1, 0, ..., 0)$, "1" being at the i-th position, $1 \leqslant i \leqslant n$. We write $\mathcal{U}(\mathbb{N}^n) = \{u_1, \ldots, u_n\}$.

For a non-empty $U \subseteq \mathcal{U}(\mathbb{N}^n)$, we denote by $U^\star$ the subtomonoid of $\mathbb{N}^n$ generated by U. We can obviously identify $U^\star$ with $\mathbb{N}^k$, where k is the number of element of U. In what follows, we will tacitly alternate between the two viewpoints.

We endow $\mathbb{N}^n$ with the natural partial order (i.e., Green's preorder $\mathcal{H}$), denoted by $\trianglelefteq$. Note that $\trianglelefteq$ is simply the component-wise order: for $(a_1, ..., a_n), (b_1, ..., b_n) \in \mathbb{N}^n$, we have

$$(a_1, ..., a_n) \trianglelefteq (b_1, ..., b_n) \quad \text{if} \quad a_1 \leqslant b_1, \ldots, a_n \leqslant b_n, \tag{1}$$

where "$\leqslant$" is the usual order of natural numbers.

A *preorder* $\preccurlyeq$ on $\mathbb{N}^n$ is a reflexive and transitive binary relation on $\mathbb{N}^n$. We write $a \prec b$ if $a \preccurlyeq b$ but not $b \preccurlyeq a$. We call $\preccurlyeq$ *total* if, for any pair $a, b \in \mathbb{N}^n$, at least one of $a \preccurlyeq b$ or $b \preccurlyeq a$ holds. Moreover, we call $\preccurlyeq$ *compatible* if, for any $a, b, c \in \mathbb{N}^n$, $a \preccurlyeq b$ implies $a + c \preccurlyeq b + c$. Finally, we call $\preccurlyeq$ *positive* if $\bar{0} \prec a$ for all $a \neq \bar{0}$.

With a preorder $\preccurlyeq$, we may associate its *symmetrisation*, which we denote by $\approx$. That is, we define $a \approx b$ if $a \preccurlyeq b$ and $b \preccurlyeq a$. The equivalence class of some a w.r.t. $\approx$ is called a $\preccurlyeq$*-class* and will be denoted by $\langle a \rangle_{\preccurlyeq}$. The quotient w.r.t. $\approx$ is denoted by $\langle \mathbb{N}^n \rangle_{\preccurlyeq}$, and its induced partial order is denoted by $\preccurlyeq$ again. Furthermore, if $\preccurlyeq$ is compatible, $\approx$ is compatible with the addition; we denote the induced operation again by $+$. Finally, if $\preccurlyeq$ is positive, the $\preccurlyeq$-class of $\bar{0}$ consists of zero alone, that is, $\langle \bar{0} \rangle_{\preccurlyeq} = \{\bar{0}\}$.

Proposition 2.6. *Let L be a p.c. tomonoid that is generated by $g_1, \ldots, g_n \in L \setminus \{0\}$, $n \geqslant 1$. Let $\varphi\colon \mathbb{N}^n \to L$ be the surjective homomorphism such that $\varphi(u_i) = g_i$, $i = 1, \ldots, n$. For $a, b \in \mathbb{N}^n$, let*

$$a \preccurlyeq b \quad \text{if} \quad \varphi(a) \leqslant \varphi(b).$$

Then $\preccurlyeq$ is a compatible, positive total preorder on $\mathbb{N}^n$, and φ induces an isomorphism between $(\langle \mathbb{N}^n \rangle_{\preccurlyeq}; \preccurlyeq, +, \{\bar{0}\})$ and $(L; \leqslant, +, 0)$.

Conversely, let $\preccurlyeq$ be a compatible, positive total preorder on $\mathbb{N}^n$, $n \geqslant 1$. Then $(\langle \mathbb{N}^n \rangle_{\preccurlyeq}; \preccurlyeq, +, \{\bar{0}\})$ is a p.c. tomonoid, which is generated by $\langle u_1 \rangle_{\preccurlyeq}, \ldots, \langle u_n \rangle_{\preccurlyeq}$.

We will call a compatible, positive total preorder on $\mathbb{N}^n$ *monomial* [Vet2]. This notion is borrowed from the theory of Gröbner bases; we recall that a monomial order can be identified with a compatible, positive total *order* on $\mathbb{N}^n$ [CLS].

By Proposition 2.6, there is a mutual correspondence between non-trivial p.c. tomonoids that are generated by at most n elements on the one hand and monomial preorders on $\mathbb{N}^n$ on the other hand. The correspondence is not one-to-one because the generators can be chosen in different ways; even repetitions are not excluded. We note that we could establish uniqueness by using the minimal set of generators [EKMMW] ordered according to $\leqslant$, and by requiring $u_1 \prec \ldots \prec u_n$. Here, however, we do not have a practical reason of doing so.

Let L be a p.c. tomonoid generated by n elements and let $\preccurlyeq$ be a monomial preorder on $\mathbb{N}^n$ such that L is isomorphic with $\langle \mathbb{N}^n \rangle_{\preccurlyeq}$. Then we say that $\preccurlyeq$ *represents* L.

Specifying finitely generated p.c. tomonoids means specifying monomial preorders. Although our topic are tomonoids, we actually focus on monomial preorders. Properties applying to p.c. tomonoids will be applied to monomial preorders as well. In particular, a monomial preorder representing a finite p.c. tomonoid is called finite as well.

It remains to say what, on the side of preorders, corresponds to the formation of quotients. The following lemma gives the immediate answer. Here, a tomonoid quotient is called *pure* if the class of the 0 is a singleton.

Lemma 2.7. *Let the monomial preorder $\preccurlyeq$ on $\mathbb{N}^n$ represent the tomonoid L. Then any pure tomonoid quotient of L is represented by a monomial preorder extending $\preccurlyeq$. Conversely, any monomial preorder on $\mathbb{N}^n$ extending $\preccurlyeq$ represents a pure tomonoid quotient of L.*

The free group generated by n elements will be identified with $\mathbb{Z}^n$. For $U \subseteq \mathcal{U}(\mathbb{N}^n)$, we denote by $U^{\star\star}$ the subgroup of $\mathbb{Z}^n$ generated by U.

We endow also $\mathbb{Z}^n$ with the component-wise order, which we denote again by $\trianglelefteqslant$. Obviously, $\mathbb{Z}^n$ then becomes a lattice-ordered group, the positive cone being $\mathbb{N}^n$. For $z \in \mathbb{Z}^n$, we write $z^+ = z \vee \bar{0}$ and $z^- = -z \vee \bar{0}$. We then have $z^+, z^- \trianglerighteqslant \bar{0}$ and $z = z^+ - z^-$.

We now introduce our main tool for the specification of monomial preorders. For proofs and further details, we refer to [Vet2].

Definition 2.8. Let $\preccurlyeq$ be a monomial preorder on $\mathbb{N}^n$. Then the set

$$C_{\preccurlyeq} \;=\; \{z \in \mathbb{Z}^n : z^- \preccurlyeq z^+\}$$

is called the *direction cone* of $\preccurlyeq$.

We note that, by the compatibility of a monomial preorder $\preccurlyeq$, we have $C_{\preccurlyeq} = \{z \in \mathbb{Z}^n : a \preccurlyeq b$ for any $a, b \in \mathbb{N}^n$ such that $b - a = z\}$.

We may characterise direction cones as follows. Let us call a k-tuple $(x_1, \ldots, x_k)$, $k \geqslant 2$, of elements of $\mathbb{Z}^n$ *addable* if

$$(x_1 + \ldots + x_k)^- + x_1 + \ldots + x_i \;\trianglerighteqslant\; \bar{0} \tag{2}$$

for all $i = 0, \ldots, k$.

Theorem 2.9. *A set $C \subseteq \mathbb{Z}^n$ is the direction cone of a monomial preorder if and only if C fulfils the following conditions:*

(C1) *Let $z \in \mathbb{N}^n$. Then $z \in C$ and, if $z \neq \bar{0}$, $-z \notin C$.*

(C2) *Let $(x_1, \ldots, x_k)$, $k \geqslant 2$, be an addable k-tuple of elements of C. Then $x_1 + \ldots + x_k \in C$.*

(C3) *Let $z \in \mathbb{Z}^n$. Then $z \in C$ or $-z \in C$.*

By Theorem 2.9, we may speak about direction cones without mentioning a preorder, referring to subsets of $\mathbb{Z}^n$ fulfilling (C1)–(C3).

A direction cone gives rise to a monomial preorder as follows.

Definition 2.10. Let $C \subseteq \mathbb{Z}^n$ be a direction cone. Let $\preccurlyeq_C$ be the smallest preorder on $\mathbb{N}^n$ such that the following holds:

(O) $a \preccurlyeq_C b$ for any $a, b \in \mathbb{N}^n$ such that $b - a \in C$.

Then we call $\preccurlyeq_C$ the preorder *induced* by C.

Lemma 2.11. *Let $C \subseteq \mathbb{Z}^n$ be a direction cone. Then $\preccurlyeq_C$ is a monomial preorder. Moreover, the direction cone of $\preccurlyeq_C$ is again C.*

For the Galois correspondence between monomial preorders and direction cones, we refer to [Vet2]. Here, we just mention the following fact.

Theorem 2.12. *Let $\preccurlyeq$ be a monomial preorder. Then $\preccurlyeq_{C_\preccurlyeq}$, the monomial preorder induced by the direction cone of $\preccurlyeq$, is contained in $\preccurlyeq$. Moreover, the direction cones of both monomial preorders coincide.*

In other words, any monomial preorder contains a monomial preorder induced by a direction cone. Let us call a tomonoid represented by a monomial preorder that is induced by a direction cone a *cone tomonoid*. Then Theorem 2.12 implies for tomonoids the following.

Theorem 2.13. *Each finitely generated p.c. tomonoid is the quotient of a cone tomonoid.*

3 Finite monomial orders and direction f-cones

Direction cones describe finitely generated p.c. tomonoids. In the finite case, however, they do not well serve this purpose; property (C3) in Theorem 2.9 implies that they are infinite. The aim of the present section is to develop a more "economical" description. In [Vet2], we have dealt with the nilpotent case; here, we proceed without this restriction.

In this section, $\preccurlyeq$ is a fixed finite monomial preorder on $\mathbb{N}^n$. As a first step, we will see what we can say about the $\preccurlyeq$-classes in general. A description of the congruences of free commutative monoids has been provided in [EiSc]; see, e.g., also [Hir]. In the presence of a total order, we can be more specific.

We start by considering the finite $\preccurlyeq$-classes. Recall that $\mathbb{N}^n$ is endowed with its natural order $\trianglelefteqslant$. We call a downwards closed non-empty subset S of $\mathbb{N}^n$ a $\trianglelefteqslant$-*ideal*.

Proposition 3.1. *Let $Z_\preccurlyeq$ be the union of all finite $\preccurlyeq$-classes. Then $Z_\preccurlyeq$ is a finite $\trianglelefteqslant$-ideal containing $\bar{0}$.*

Moreover, let $a \in \mathbb{N}^n$. Then $a \in Z_\preccurlyeq$ if and only if $a \prec b$ for any $b \rhd a$. In this case, $c \prec a$ for any $c \lhd a$.

Proof. We first show the second part. Let $a, b \in \mathbb{N}^n$ be such that $a \lhd b$ and $a \approx b$. Putting $d = b - a$, it then follows $a \approx b = a + d \approx b + d = a + 2\,d \approx \ldots$, that is, $\langle a \rangle_\preccurlyeq$ is infinite. Similarly, let $a, c \in \mathbb{N}^n$ be such that $c \lhd a$ and $a \approx c$. Then it follows again that $\langle a \rangle_\preccurlyeq$ is infinite.

Let now $a \in \mathbb{N}^n$ be such that $\langle a \rangle_\preccurlyeq$ is infinite. For each $i = 1, \ldots, n$, there is an $n_i \geqslant 1$ such that $n_i\, u_i \approx (n_i + 1)\, u_i$, because otherwise $u_i \prec 2u_i \prec 3u_i \prec \ldots$ and this would mean that there are infinitely many $\preccurlyeq$-classes. Hence the equivalence classes of elements of the form $\Sigma_i k_i\, u_i$ such that $k_i \leqslant n_i$ for each i and $k_i = n_i$ for at least one i cover a cofinite subset of $\mathbb{N}^n$. We conclude that a is equivalent to some c such that $c \approx c + u_i$ for some i. It follows $a \lhd a + u_i \approx c + u_i \approx c \approx a$. The proof of the second part is complete.

To show the first statement, assume that $\langle a \rangle_{\preccurlyeq}$ is infinite and $b \unrhd a$. Then we have seen that $a \approx a + u_i$ for some $1 \leqslant i \leqslant n$. It follows that $b \approx b + u_i$ as well, hence also $\langle b \rangle_{\preccurlyeq}$ is infinite. We have shown that $Z_{\preccurlyeq}$ is a $\unlhd$-ideal. Clearly, $Z_{\preccurlyeq}$ is finite because there are, by assumption, only finitely many $\preccurlyeq$-classes. Finally, $\langle \bar{0} \rangle_{\preccurlyeq} = \{\bar{0}\}$ because $\preccurlyeq$ is positive, hence $\bar{0} \in Z_{\preccurlyeq}$. □

We see that the union of the finite $\preccurlyeq$-classes $Z_{\preccurlyeq}$ is a finite $\unlhd$-ideal and the idea is natural to describe $\preccurlyeq$ separately on $Z_{\preccurlyeq}$ and its complement. To this end, we might want to define a finitary analogue of a direction cone by considering $\preccurlyeq$ only on $Z_{\preccurlyeq}$. To specify the rest of $\preccurlyeq$ we should moreover make use of the fact that each infinite $\preccurlyeq$-class is the finite union of semilinear sets [EiSc, Hir].

We will indeed roughly follow these ideas, but in order to take into account properly all relevant aspects, details will differ. Namely, we will describe finite monomial preorders by a triple: the first constituent corresponds to the tomonoid's Archimedean classes; the second one is a finite $\unlhd$-ideal of $\mathbb{N}^n$ that properly includes the finite $\preccurlyeq$-classes and determines to a good extent also the infinite $\preccurlyeq$-classes; and the third one describes the preorder itself.

We begin by defining the partition of $\mathbb{N}^n$ into Archimedean classes. We use common definitions; see, e.g., [Fuc]; but we apply them directly to the preorder $\preccurlyeq$ rather than to the tomonoid represented by it.

For $a, b \in \mathbb{N}^n$, we put

$$a \prec\!\!\prec b \quad \text{if } \; k\, a \prec b \text{ for all } k \geqslant 1,$$

and two elements a and b of $\mathbb{N}^n$ are called *Archimedean equivalent* if neither $a \prec\!\!\prec b$ nor $b \prec\!\!\prec a$. Obviously, this is the case if and only if either $a \preccurlyeq b$ and $b \preccurlyeq k\, a$ for some $k \geqslant 1$, or the other way round.

The relation $\prec\!\!\prec$ is uniquely determined by its restriction to the generators $u_1, \ldots, u_n$. Accordingly, we define our first constituent as follows. By an *ordered partition* of a set U, we mean a finite sequence of pairwise disjoint and jointly exhaustive non-empty subsets of U.

Definition 3.2. We define

$$A_{\preccurlyeq} \;=\; (U_1, \ldots, U_m)$$

to be the ordered partition of $\mathcal{U}(\mathbb{N}^n)$ such that, for each $u, v \in \mathcal{U}(\mathbb{N}^n)$, $u \prec\!\!\prec v$ if and only if $i < j$, where $u \in U_i$ and $v \in U_j$. We call $A_{\preccurlyeq}$ the *generator partition* of $\preccurlyeq$.

Moreover, we call $\preccurlyeq$ *Archimedean* if $m = 1$, and *non-Archimedean* if $m \geqslant 2$.

Given an ordered partition $(U_1, \ldots, U_m)$, we will write in the sequel $U_{\geqslant k}$, where $1 \leqslant k \leqslant m$, to denote the union $U_k \cup \ldots \cup U_m$, and similarly for $U_{<k}$ and $U_{\leqslant k}$.

The generator partition of $\preccurlyeq$ obviously determines the relation $\prec\!\!\prec$ on $\mathbb{N}^n$ uniquely.

Definition 3.3. Let $A = (U_1, \ldots, U_m)$ be an ordered partition of $\mathcal{U}(\mathbb{N}^n)$. For $a, b \in \mathbb{N}^n$, we define

$$a \prec\!\!\prec_A b$$

if there is an $1 \leqslant k \leqslant m$ such that $a \in U_{<k}{}^\star$ and $u \trianglelefteq b$ for some $u \in U_{\geqslant k}$.

Proposition 3.4. *Let $A = A_\preccurlyeq$ be the generator partition of $\preccurlyeq$. Then $a \prec\!\!\prec b$ if and only if $a \prec\!\!\prec_A b$.*

Second, we turn to the question if we can define a finite $\trianglelefteq$-ideal S of $\mathbb{N}^n$ with the following properties: (a) S contains all finite $\preccurlyeq$-classes, (b) S contains at least one element of each infinite $\preccurlyeq$-class, and (c) there is a way to tell for an arbitrary element outside S that it is in the same $\preccurlyeq$-class as a certain element of S. We will see that this is indeed possible.

We need some more notation. Given $a \in \mathbb{N}^n \setminus \{\bar{0}\}$, we denote by $s(a)$ the smallest $i \in \{1, \ldots, m\}$ such that $u \trianglelefteq a$ for some $u \in U_i$. Moreover, we write $u \overset{\min}{\trianglelefteq} a$ if $u \trianglelefteq a$ and $u \in U_{s(a)}$.

Definition 3.5. We call

$$S_\preccurlyeq \;=\; \{a \in \mathbb{N}^n\colon\; a = \bar{0}, \text{ or } a - u \prec a \text{ for some } u \overset{\min}{\trianglelefteq} a\}$$

the *support* of $\preccurlyeq$.

Let us call a $\trianglelefteq$-ideal of $\mathbb{N}^n$ *non-degenerate* if it contains all $u \in \mathcal{U}(\mathbb{N}^n)$.

Lemma 3.6. *$S_\preccurlyeq$ is a finite, non-degenerate $\trianglelefteq$-ideal of $\mathbb{N}^n$.*

Proof. Let $a \notin S_\preccurlyeq$. Then $a \neq \bar{0}$ and $a - u \approx a$ for any $u \overset{\min}{\trianglelefteq} a$. Let $b \trianglerighteq a$ and $v \overset{\min}{\trianglelefteq} b$. Then $b - v \approx b$ as well, as seen in each of the following cases:

Case 1. Let $s(b) < s(a)$. Then $v \prec u$ for any $u \overset{\min}{\trianglelefteq} a$ and thus $b \approx b - u \preccurlyeq b - v \preccurlyeq b$.

Case 2. Let $s(b) = s(a)$ and $v \overset{\min}{\trianglelefteq} a$. Then $a - v \approx a$ implies $b - v \approx b$.

Case 3. Let $s(b) = s(a)$ and $a \trianglelefteq b - v$. Because $a \approx a + ku$ for any $k \geqslant 1$ and u and v are Archimedean equivalent, we have $a \approx a + v$. Hence $b - v \approx (b - v) + v = b$.

We conclude that $b \notin S_\preccurlyeq$ and hence that $S_\preccurlyeq$ is a $\trianglelefteq$-ideal.

As $\preccurlyeq$ is positive, we have $0 \prec u$ and consequently $u \in S_\preccurlyeq$ for all $u \in \mathcal{U}(\mathbb{N}^n)$. □

Given an ordered partition $A = (U_1, \ldots, U_m)$ of $\mathbb{N}^n$, let us associate the following sets with a finite, non-degenerate $\trianglelefteq$-ideal $S \subseteq \mathbb{N}^n$. Here, the value $s(a)$ is understood with reference to A as specified above. We call

$$\begin{aligned} \mathring{S} &= \{a \in S\colon\; a = \bar{0}, \text{ or } a + u \in S \text{ for all } u \in U_j \text{ such that } j \leqslant s(a)\}, \\ \partial S &= S \setminus \mathring{S} \end{aligned}$$

the *core* and the *boundary* of S, respectively. Moreover, let $a \in \partial S$. This means $a \in S$ but $a + u \notin S$ for some $u \in U_j$ such that $j \leqslant s(a)$. We call

$$\sigma_S(a) \;=\; a + U_{\leqslant j}{}^\star$$

the *segment* of a, where $j \in \{1, \ldots, s(a)\}$ is largest such that $a + u \notin S$ for some $u \in U_j$. Note that S and the segments of all $a \in \partial S$ cover the whole $\mathbb{N}^n$.

Proposition 3.7. (i) *$\mathring{S}_{\preccurlyeq}$ is the union of the finite $\preccurlyeq$-classes.*

(ii) *Let B be an infinite $\preccurlyeq$-class. Then B has a non-empty intersection with $S_{\preccurlyeq}$; all $\trianglelefteq$-minimal elements of B are contained in $B \cap S_{\preccurlyeq}$, which in turn is a subset of $\partial S_{\preccurlyeq}$. Moreover,*

$$B \;=\; \bigcup_{a \in B \cap \partial S_{\preccurlyeq}} \sigma_{S_{\preccurlyeq}}(a). \tag{3}$$

Proof. (i) Let $a \in \mathbb{N}^n$ be such that $\langle a \rangle_{\preccurlyeq}$ is finite. If $a = \bar{0}$, we have $a \in \mathring{S}_{\preccurlyeq}$ by definition. Assume $a \neq \bar{0}$. Then $a \in S_{\preccurlyeq}$ by Proposition 3.1. Moreover, for any $u \in U_j$ such that $j \leqslant s(a)$, we have $a \prec a + u$ again by Proposition 3.1 and it follows $a + u \in S_{\preccurlyeq}$ as well, that is, $a \in \mathring{S}_{\preccurlyeq}$.

Let now $a \in \mathbb{N}^n$ be such that $\langle a \rangle_{\preccurlyeq}$ is infinite. Then, by Proposition 3.1, there is some $b \vartriangleright a$ such that $a \approx b$. It follows that there is an $u \in U_1$ such that $a \approx a + u$. Let v be, w.r.t. $\preccurlyeq$, the largest element of U_1. Then $a \approx a + v$ because u and v are Archimedean equivalent. If $a \in \mathring{S}_{\preccurlyeq}$, we would have $a + v \in S_{\preccurlyeq}$ and hence there would be a $u' \in U_1$ such that $u' \trianglelefteq a + v$ and $a \preccurlyeq (a + v) - u' \prec a + v$, a contradiction. Thus $a \notin \mathring{S}_{\preccurlyeq}$.

(ii) Let b be a $\trianglelefteq$-minimal element of B. Then $b \neq \bar{0}$ by Lemma 3.1 and hence $b - u \prec b$ for all $u \overset{\min}{\trianglelefteq} b$. In particular, $b \in B \cap S_{\preccurlyeq}$. By part (i), $B \cap S_{\preccurlyeq} \subseteq \partial S_{\preccurlyeq}$.

It remains to prove (3). To see the "$\subseteq$" part, let $b \in B$. If $b \in \partial S_{\preccurlyeq}$, we have $b \in \sigma_{S_{\preccurlyeq}}(b)$. Otherwise, b is not a $\trianglelefteq$-minimal element of B, hence $b - u \approx b$ for some $u \in \mathcal{U}(\mathbb{N}^n)$ such that $u \trianglelefteq b$. W.l.o.g., we may assume that $u \in U_{s(b)}$ in this case. If $b - u \notin \partial S_{\preccurlyeq}$, we may continue arguing in the same manner, to conclude that there is an $a \trianglelefteq b$ such that $a \in B \cap \partial S_{\preccurlyeq}$ and $b \in \sigma_{S_{\preccurlyeq}}(a)$.

For the "$\supseteq$" part, let $a \in B \cap \partial S_{\preccurlyeq}$. Let $j \leqslant s(a)$ be largest such that, for some $u \in U_j$, we have $a + u \notin S_{\preccurlyeq}$. Then $a \approx a + u$ and we conclude that $a \approx a + v$ for any $v \in U_j$ and hence even for any $v \in U_i$ such that $i \leqslant j$. This means that $b \in \sigma_{S_{\preccurlyeq}}(a)$ implies $b \approx a$, that is, $b \in B$. □

We observe from Proposition 3.7 that $S_{\preccurlyeq}$ has the three properties (a)–(c) indicated above. In fact, in case of (a) and (b), this is immediate. To verify (c), let $b \notin S_{\preccurlyeq}$; then we can, by Proposition 3.7, determine as follows an element $a \in \partial S_{\preccurlyeq}$ such that $a \approx b$. We generate a sequence $b = b_0 \trianglerighteq b_1 \trianglerighteq \ldots$ such that, for each $i \geqslant 1$, $b_i = b_{i-1} - u$ for some $u \overset{\min}{\trianglelefteq} b_{i-1}$. We eventually arrive at an element $a = b_k \in S_{\preccurlyeq}$; then $a \in \partial S_{\preccurlyeq}$ and $b \in \sigma_{S_{\preccurlyeq}}(a)$, that is, $a \approx b$.

We conclude in particular that the generator classes $A_\preccurlyeq$, the support $S_\preccurlyeq$, and $\preccurlyeq$ restricted to $S_\preccurlyeq$ uniquely determine the whole preorder $\preccurlyeq$. There is hence a natural way of defining the third and main constituent.

For a $\trianglelefteq$-ideal $S \subseteq \mathbb{N}^n$, let us write

$$\mathcal{D}(S) = \{z \in \mathbb{Z}^n : z^-, z^+ \in S\}.$$

Definition 3.8. We define

$$F_\preccurlyeq \;=\; \{z \in \mathcal{D}(S_\preccurlyeq) : z^- \preccurlyeq z^+\}.$$

Furthermore, we call the triple $\mathcal{C}_\preccurlyeq = (A_\preccurlyeq, S_\preccurlyeq, F_\preccurlyeq)$ the *direction f-cone* of $\preccurlyeq$.

Here, the "f" stands for "finite". We note that our present definition differs from the one given in [Vet2], where we have dealt with a more special case.

The characterisation of direction f-cones is not as easy as in the case of direction cones. In the sequel, when speaking about a direction f-cone without reference to a preorder, we mean the direction f-cone of some finite monomial preorder.

4 A way of constructing finite monomial preorders

The previous section was devoted to the congruences on $\mathbb{N}^n$ associated with finite p.c. tomonoids. Based on the results obtained, we have introduced the notion of a direction f-cone as a means of describing finite monomial preorders.

We will see next that direction f-cones fulfil properties similarly to those of direction cones; cf. Theorem 2.9. Even though we cannot characterise direction f-cones in the same simple way as direction cones, we will see that any triple (A, S, F) fulfilling the stated properties gives rise to a finite monomial preorder. Moreover, any finite monomial preorder contains a preorder obtained in this way.

Proposition 4.1. *Let $\preccurlyeq$ be a finite monomial preorder. Then the direction f-cone (A, S, F) of $\preccurlyeq$ has the following properties:*

(Cf1) *For each $z \in \mathcal{D}(S)$, $z \trianglerighteq 0$ implies $z \in F$ and, if $z \neq 0$, $-z \notin F$.*

(Cf2) *Let $(x_1, \ldots, x_k)$, $k \geqslant 2$, be an addable k-tuple of elements of F whose sum is in $\mathcal{D}(S)$. Then $x_1 + \ldots + x_k \in F$.*

(Cf3) *For each $z \in \mathcal{D}(S)$, either $z \in F$ or $-z \in F$.*

(Cf4) *Let $a, b \in S$ be such that $a \prec\!\!\prec_A b$. Then $a - b \notin F$.*

Proof. In case of (Cf1)–(Cf3), the arguments coincide with those of the proof of Theorem 2.9; see [Vet2].

To see (Cf4), assume that $a, b \in S$ are such that $a \prec\!\!\prec_A b$. Then $a \prec\!\!\prec b$ by Proposition 3.4 and consequently $a \prec b$. But $a - b \in F$ would imply $b \preccurlyeq a$. $\square$

Our aim is to show the conditions (Cf1)–(Cf4) of Proposition 4.1 are strong enough to make the construction of monomial preorders possible.

Definition 4.2. Let $\mathcal{C} = (A, S, F)$, where A is an ordered partition of $\mathcal{U}(\mathbb{N}^n)$, S is a finite, non-degenerate $\trianglelefteq$-ideal of $\mathbb{N}^n$, and $F \subseteq \mathcal{D}(S)$. Let $\preccurlyeq_{\mathcal{C}}$ be the smallest preorder on $\mathbb{N}^n$ such that the following holds:

(O1) $a \preccurlyeq_{\mathcal{C}} b$ for any $a, b \in \mathbb{N}^n$ such that $b - a \in F$.

(O2) $a \preccurlyeq_{\mathcal{C}} b$ and $b \preccurlyeq_{\mathcal{C}} a$ for any $a \in \partial S$ and $b \in \sigma_S(a)$.

Then we call $\preccurlyeq_{\mathcal{C}}$ the preorder *induced* by $\mathcal{C}$.

Lemma 4.3. *Let $\mathcal{C} = (A, S, F)$, where A is an ordered partition of $\mathcal{U}(\mathbb{N}^n)$, S is a finite, non-degenerate $\trianglelefteq$-ideal of $\mathbb{N}^n$, and $F \subseteq \mathcal{D}(S)$. Assume that $\mathcal{C}$ fulfils properties* (Cf1)–(Cf4). *Then $\preccurlyeq_{\mathcal{C}}$ is a finite monomial preorder.*

Proof. By construction, $\preccurlyeq_{\mathcal{C}}$ is a preorder. $\preccurlyeq_{\mathcal{C}}$ is finite because the union of S and $\sigma_S(a)$, $a \in \partial S$, covers $\mathbb{N}^n$ and by (O2) each segment is included in a $\preccurlyeq_{\mathcal{C}}$-class. $\preccurlyeq_{\mathcal{C}}$ is total because, by (Cf3), any two elements of S are comparable and each infinite class has a non-empty intersection with S.

Our next aim is to show the translation invariance of $\preccurlyeq_{\mathcal{C}}$. Let $a, b, t \in \mathbb{N}^n$. Assume that $a \preccurlyeq_{\mathcal{C}} b$ holds according to prescription (O1). Then $b - a \in F$ and hence $a + t \preccurlyeq_{\mathcal{C}} b + t$ by (O1) as well.

Assume moreover that $a \in \partial S$ and $b \in \sigma_S(a)$ and thus $a \approx_{\mathcal{C}} b$ according to prescription (O2). Let $j \leqslant s(a)$ be such that $\sigma_S(a) = a + {U_{\leqslant j}}^{\star}$. Then we have $b = a + c$ for some $c \in {U_{\leqslant j}}^{\star}$. Let furthermore $t = t_1 + t_2$, where $t_1 \in U_{>j}$ and $t_2 \in U_{\leqslant j}$. We distinguish two cases.

Case 1. Let $a' = a + t_1 \in \partial S$. Then $a + u \notin S$ implies $a' + u \notin S$ for any $u \in U_{\leqslant j}$. Moreover, $j \leqslant s(a)$ and $j \leqslant s(t_1)$, hence $j \leqslant s(a')$. It follows $a' + {U_{\leqslant j}}^{\star} \subseteq \sigma_S(a')$. We conclude $a + t = a' + t_2 \in \sigma_S(a')$ and $b + t = a' + c + t_2 \in \sigma_S(a')$.

Case 2. Let $a + t_1 \notin S$. Let $a' \lhd a + t_1$ such that $a + t_1 \in \sigma_S(a')$. Then $s(a + t_1) \geqslant j$ and $(a + t_1) - a' \neq \bar{0}$. Hence $s((a + t_1) - a') \geqslant j$ and we have again $a' + U_{\leqslant j} \subseteq \sigma_S(a')$. We conclude as in Case 1 that $a + t, b + t \in \sigma_S(a')$.

We have shown $a + t \approx_{\mathcal{C}} b + t$. The proof of the translation invariance is complete.

It remains to show that $\preccurlyeq_{\mathcal{C}}$ is positive. Indeed, if $a \preccurlyeq_{\mathcal{C}} \bar{0}$ holds according to (O1), $a = \bar{0}$ by (Cf1). Moreover, $a \preccurlyeq_{\mathcal{C}} \bar{0}$ cannot hold according to (O2) because $\bar{0} \notin \partial S$. It follows that $\preccurlyeq_{\mathcal{C}}$ is positive. □

The direction f-cone of the preorder obtained in this way does not necessarily coincide with the triple from which we started. We can say the following.

Lemma 4.4. *Let $\mathcal{C} = (A, S, F)$, where A is an ordered partition of $\mathcal{U}(\mathbb{N}^n)$, S is a finite, non-degenerate $\trianglelefteq$-ideal of $\mathbb{N}^n$, and $F \subseteq \mathcal{D}(S)$. Assume that $\mathcal{C}$ fulfils properties* (Cf1)–(Cf4). *Then the direction f-cone of $\preccurlyeq_{\mathcal{C}}$ has the following properties:*

(i) $A_{\preccurlyeq_{\mathcal{C}}} = A$.

(ii) $S_{\preccurlyeq_{\mathcal{C}}} \subseteq S$.

(iii) $F \cap \mathcal{D}(S_{\preccurlyeq_{\mathcal{C}}}) \subseteq F_{\preccurlyeq_{\mathcal{C}}}$. *Moreover, if* $z \in F_{\preccurlyeq_{\mathcal{C}}}$ *and* $z^- \in \mathring{S}_{\preccurlyeq_{\mathcal{C}}}$ *or* $z^+ \in \mathring{S}_{\preccurlyeq_{\mathcal{C}}}$*, then* $z \in F \cap \mathcal{D}(S_{\preccurlyeq_{\mathcal{C}}})$.

Proof. (i) We have to show that A is the generator partition of $\preccurlyeq_{\mathcal{C}}$. Let $u \in U_i$ and $v \in U_j$ and assume $i < j$. We claim that $u \prec\!\!\prec_{\mathcal{C}} v$ then. Indeed, otherwise, there would be $a, b \in \mathbb{N}^n$ such that $a \prec\!\!\prec_A b$ and $b \preccurlyeq_{\mathcal{C}} a$ according to prescription (O1) or (O2). However, (O1) is not applicable because $a - b \notin F$ by (Cf4). Moreover, if $a \in \partial S$, then b cannot be in $\sigma_S(a)$, and vice versa; hence also (O2) does not apply.

Let now $1 \leqslant i \leqslant m$ and $u, v \in U_i$. Then $u - v, v - u \in \mathcal{D}(S)$ because $\mathcal{U}(\mathbb{N}^n) \subseteq S$. By (Cf3), one of these differences is in F; w.l.o.g., assume that $v - u \in F$ and hence $u \preccurlyeq_{\mathcal{C}} v$. Let furthermore $k, l \geqslant 1$ be greatest such that $ku, lv \in S$, respectively. Then $ku, lv \in \partial S$ and $ku + lv \in \sigma_S(ku), \sigma_S(lv)$, that is, $ku \approx_{\mathcal{C}} lv$ and it follows $v \preccurlyeq_{\mathcal{C}} ku$. Hence u and v are Archimedean equivalent. The proof is complete that the generator partition of $\preccurlyeq_{\mathcal{C}}$ is A.

(ii) Let $a \notin S$ and $u \overset{\min}{\trianglelefteq} a$. Let $a' \trianglelefteq a - u$ be such that $a' \in \partial S$ and $a - u \in \sigma_S(a')$. Then also $a \in \sigma_S(a')$ and hence $a \approx_{\mathcal{C}} a - u$. It follows that $a \notin S_{\preccurlyeq_{\mathcal{C}}}$.

(iii) For any $z \in F$ we have $z^- \preccurlyeq_{\mathcal{C}} z^+$ by (O1). Hence $z \in F_{\preccurlyeq_{\mathcal{C}}}$, provided that $z \in \mathcal{D}(S_{\preccurlyeq_{\mathcal{C}}})$. Hence $F \cap \mathcal{D}(S_{\preccurlyeq_{\mathcal{C}}}) \subseteq F_{\preccurlyeq_{\mathcal{C}}}$.

Moreover, let $z \in F_{\preccurlyeq_{\mathcal{C}}}$ such that z^- or z^+ is in $\mathring{S}_{\preccurlyeq_{\mathcal{C}}}$. Then $z \in \mathcal{D}(S_{\preccurlyeq_{\mathcal{C}}}) \subseteq \mathcal{D}(S)$ and $z^- \preccurlyeq_{\mathcal{C}} z^+$. Hence there are $a_0, \ldots, a_k \in \mathbb{N}^n$ such that $z^- = a_0,\ z^+ = a_k$, and for each $i = 0, \ldots, k-1,\ a_i \preccurlyeq_{\mathcal{C}} a_{i+1}$ holds according to prescription (O1) or (O2). Assume now that $z \notin F$. Then $-z \in F$ by (Cf3) and consequently $z^+ \preccurlyeq_{\mathcal{C}} z^-$. Hence $a_0, \ldots, a_k$ all belong to the same finite $\preccurlyeq_{\mathcal{C}}$-class and (O1) applies in each case. By (Cf2), $z \in F$, a contradiction. The last assertion follows as well. □

We have constructed finite monomial preorders from triples (A, S, F) subject to conditions (Cf1)-(Cf4). Let us next see to which monomial preorders we are led when we start from a direction f-cone. The situation is then similar to the case of direction cones; cf. Theorem 2.12.

Theorem 4.5. *Let* $\preccurlyeq$ *be a finite monomial preorder. Then* $\preccurlyeq_{\mathcal{C}_{\preccurlyeq}}$*, the monomial preorder induced by the direction f-cone of* $\preccurlyeq$*, is a monomial preorder contained in* $\preccurlyeq$*. Moreover, the direction f-cones of both preorders coincide.*

Proof. Let $\mathcal{C} = \mathcal{C}_{\preccurlyeq} = (A, S, F)$. By Proposition 4.1 and Lemma 4.3, $\preccurlyeq_{\mathcal{C}}$ is a monomial preorder. By the definition of F and by Proposition 3.7, $\preccurlyeq_{\mathcal{C}} \subseteq \preccurlyeq$.

By Lemma 4.3, $A_{\preccurlyeq_{\mathcal{C}}} = A$ and $S_{\preccurlyeq_{\mathcal{C}}} \subseteq S$. From $A_{\preccurlyeq_{\mathcal{C}}} = A$ and $\preccurlyeq_{\mathcal{C}} \subseteq \preccurlyeq$ it also follows $S \subseteq S_{\preccurlyeq_{\mathcal{C}}}$, that is, $S = S_{\preccurlyeq_{\mathcal{C}}}$.

Let $z \in \mathcal{D}(S)$. Assume that $z \in F$. Then $z^- \preccurlyeq_{\mathcal{C}} z^+$ by (O1) and hence $z \in F_{\preccurlyeq_{\mathcal{C}}}$. Conversely, let $z \in F_{\preccurlyeq_{\mathcal{C}}}$. Then $z^- \preccurlyeq_{\mathcal{C}} z^+$ and hence $z^- \preccurlyeq z^+$, that is, $z \in F$. We conclude that $F = F_{\preccurlyeq_{\mathcal{C}}}$. □

Let us call a p.c. tomonoid that is represented by a monomial preorder induced by a direction f-cone an *f-cone tomonoid*. Then Theorem 4.5 has the following corollary.

Theorem 4.6. *Each finite p.c. tomonoid is the quotient of an f-cone tomonoid.*

In contrast to Theorem 2.13, this theorem is not very helpful as regards the construction of finite p.c. tomonoids, simply because we cannot characterise direction f-cones similarly to direction cones. However, we may combine our results to indicate a generally applicable way of constructing finite monomial preorders.

Theorem 4.7. *Let $\mathcal{C} = (A, S, F)$, where A is an ordered partition of $\mathcal{U}(\mathbb{N}^n)$, S is a finite, non-degenerate $\trianglelefteq$-ideal of $\mathbb{N}^n$, and $F \subseteq \mathcal{D}(S)$. Assume that $\mathcal{C}$ fulfils properties* (Cf1)–(Cf4). *Then $\preccurlyeq_{\mathcal{C}}$ is a finite monomial preorder.*

Moreover, any finite monomial preorder is an extension of a monomial preorder arising in this way.

Proof. By Lemma 4.3, $\preccurlyeq_{\mathcal{C}}$ is a finite monomial preorder.

Let $\preccurlyeq$ be an arbitrary finite monomial preorder. By Proposition 4.1, the direction f-cone $\mathcal{C}$ of $\preccurlyeq$ fulfils (Cf1)–(Cf4) and by Theorem 4.5, $\preccurlyeq$ extends the finite monomial preorder $\preccurlyeq_{\mathcal{C}}$. □

5 The co-Archimedean subcone of a direction f-cone

Direction f-cones give rise to finite monomial preorders, which in turn represent finite p.c. tomonoids. Moreover, any finite p.c. tomonoid is a quotient of a tomonoid arising in this way.

Characterising direction f-cones, however, turns out to be difficult. An exception is the Archimedean case. If the monomial preorder is Archimedean, the represented tomonoid is nilpotent and we can proceed as shown in our previous paper [Vet2].

The present section is based on the idea of describing a direction f-cone not as a whole, but relative to a smaller one. Namely, if there are at least two Archimedean classes, we restrict the direction f-cone to the submonoid of $\mathbb{N}^n$ generated by all those $u \in \mathcal{U}(\mathbb{N}^n)$ that do not belong to the smallest Archimedean class. The result is a direction f-cone again and we may describe the original direction f-cone relatively to it.

In this section, $\mathcal{C} = (A, S, F)$ is a fixed direction cone. We denote by $\preccurlyeq \; = \; \preccurlyeq_{\mathcal{C}}$ its induced finite monomial preorder on $\mathbb{N}^n$. Let $A = (U_1, \ldots, U_m)$. We will furthermore assume that $m \geqslant 2$, that is, $\preccurlyeq$ is non-Archimedean. We write $U_a = U_1$ to denote the first element of the generator partition and $U_r = U_2 \cup \ldots \cup U_m$ to denote the union of the remaining ones.

Definition 5.1. We call the triple $\mathcal{C}_r = (A_r, S_r, F_r)$, where

$$A_r = (U_2, \ldots, U_m), \qquad S_r = S \cap U_r{}^\star, \qquad F_r = F \cap \mathcal{D}(S_r),$$

the *co-Archimedean subcone* of (A, S, F).

We will consider three different monomial preorders on $U_r{}^\star$. To begin with, let $\preccurlyeq^r$ be the restriction of $\preccurlyeq$ to $U_r{}^\star$. The following proposition shows what the transition from $\preccurlyeq$ to $\preccurlyeq^r$ means for the represented tomonoid: we are led to the subtomonoid generated by those generators that do not belong to the smallest Archimedean class.

Proposition 5.2. *$\preccurlyeq^r$ is a monomial preorder and the tomonoid $(\langle U_r{}^\star \rangle_{\preccurlyeq^r}; \preccurlyeq^r, +, \{\bar{0}\})$ is isomorphic with the subtomonoid of $(\langle \mathbb{N}^n \rangle_{\preccurlyeq}; \preccurlyeq, +, \{\bar{0}\})$ generated by $\langle u \rangle_{\preccurlyeq}$, $u \in U_r$.*

Proof. It is clear that $\preccurlyeq^r$ is a positive, compatible, total preorder, that is, a monomial preorder.

Furthermore, $(U_r{}^\star; +, \bar{0})$ is the submonoid of $(\mathbb{N}^n; +, \bar{0})$ generated by U_r and, for any $a, b \in U_r{}^\star$, $a \preccurlyeq^r b$ if and only if $a \preccurlyeq b$. We conclude that we can define $\iota \colon \langle U_r{}^\star \rangle_{\preccurlyeq^r} \to \langle U_r{}^\star \rangle_{\preccurlyeq}, \;\; \langle a \rangle_{\preccurlyeq^r} \mapsto \langle a \rangle_{\preccurlyeq}$ and that ι is an injective and surjective homomorphism of monoids. Moreover, for any $a, b \in U_r{}^\star$ we have $\langle a \rangle_{\preccurlyeq^r} \preccurlyeq^r \langle b \rangle_{\preccurlyeq^r}$ if and only if $\langle a \rangle_{\preccurlyeq} \preccurlyeq \langle b \rangle_{\preccurlyeq}$, that is, ι is actually an isomorphism of tomonoids. □

The reason for introducing $\preccurlyeq^r$ is to show that (A_r, S_r, F_r) is in fact a direction f-cone.

Lemma 5.3. *(A_r, S_r, F_r) is the direction f-cone of $\preccurlyeq^r$.*

Proof. For $a, b \in U_r{}^\star$, we have $a \lll^r b$ if and only if $a \lll b$. Hence the Archimedean classes of $\preccurlyeq^r$ in $U_r{}^\star$ are those of $\preccurlyeq$ in $\mathcal{U}(\mathbb{N}^n)$ except for U_a. That is, $A_{\preccurlyeq^r} = A_r$.

Let $a \in U_r{}^\star \setminus \{\bar{0}\}$. Then, for any $u \in \mathcal{U}(\mathbb{N}^n)$, we have that $u \overset{\min}{\trianglelefteq} a$ holds in $\mathbb{N}^n$ w.r.t. A if and only if $u \in U_r$ and $u \overset{\min}{\trianglelefteq} a$ holds in $U_r{}^\star$ w.r.t. A_r. Consequently, $a \in S_{\preccurlyeq^r}$ iff $a - u \prec^r a$ for some $u \in U_r$ such that $u \overset{\min}{\trianglelefteq} a$ iff $a - u \prec a$ for some $u \in \mathcal{U}(\mathbb{N}^n)$ such that $u \overset{\min}{\trianglelefteq} a$ iff $a \in S_{\preccurlyeq} = S$. Hence $S_{\preccurlyeq^r} = S \cap U_r{}^\star = S_r$.

Note next that $\mathcal{D}(S_r) = \mathcal{D}(S \cap U_r{}^\star) = \mathcal{D}(S) \cap U_r{}^{\star\star}$. Let $z \in U_r{}^{\star\star}$. Then $z \in F_{\preccurlyeq^r}$ iff $z \in \mathcal{D}(S_r)$ and $z^- \preccurlyeq^r z^+$ iff $z \in \mathcal{D}(S)$ and $z^- \preccurlyeq z^+$ iff $z \in F$. Hence $F_{\preccurlyeq^r} = F \cap U_r{}^{\star\star} = F \cap \mathcal{D}(S) \cap U_r{}^{\star\star} = F \cap \mathcal{D}(S_r) = F_r$. □

Let us next consider the monomial preorder induced by (A_r, S_r, F_r); we denote it by $\preccurlyeq_r$.

Lemma 5.4. *$\preccurlyeq_r$ is contained in $\preccurlyeq^r$.*

Proof. This holds by Lemma 5.3 and Theorem 4.5. □

A third monomial preorder on $U_r{}^\star$ is constructed as follows. Let us define

$$\bar{F}_r = \{z \in \mathcal{D}(S_r) \colon z^- + f \preccurlyeq z^+ + g \text{ for some } f, g \in U_a{}^\star\}.$$

Proposition 5.5. *The triple* $(A_r, S_r, \bar{F}_r)$ *fulfils the properties* (Cf1)–(Cf4).

Proof. Note first that $z \in \bar{F}_r$ if and only if $z \in \mathcal{D}(S) \cap {U_r}^{\star\star}$ and $z^- + e \preccurlyeq z^+ + e$. Here, e is a $\preccurlyeq$-maximal element of ${U_a}^\star$.

Ad (Cf1): Let $z \in \mathcal{D}(S_r)$ such that $z \trianglerighteq \bar{0}$. Obviously, $z \in \bar{F}_r$. Assume $z \neq \bar{0}$ and $-z \in \bar{F}_r$. Then $z + f \preccurlyeq g$ for some $f, g \in {U_a}^\star$, in contradiction to the fact that $g \lll z + f$.

Ad (Cf2): Let $x_1, \ldots, x_k \in \bar{F}_r$, $k \geqslant 2$, be addable and let $z = x_1 + \ldots + x_k \in \mathcal{D}(S_r)$. Then $z^- + e \preccurlyeq z^- + x_1 + e \preccurlyeq \ldots \preccurlyeq z^- + x_1 + \ldots + x_k + e = z^+ + e$ and it follows $z \in \bar{F}_r$.

Ad (Cf3): We have $F_r = F \cap \mathcal{D}(S_r) \subseteq \bar{F}_r$.

Ad (Cf4): Let $a, b \in S_r$ be such that $a \lll_{A_r} b$. Then $a + e \lll_A b + e$ and thus $a + e \prec b + e$. It follows $(a - b)^+ + e \prec (a - b)^- + e$, hence $a - b \notin \bar{F}_r$. □

Let us denote the monomial preorder induced by $(A_r, S_r, \bar{F}_r)$ by $\bar{\preccurlyeq}_r$.

Lemma 5.6. *Let* $a, b \in {U_r}^\star$. *Then* $a \mathrel{\bar{\preccurlyeq}_r} b$ *if and only if, for some* $f, g \in {U_a}^\star$, $a + f \preccurlyeq b + g$.

Proof. "Only if" part: Assume first that $a, b \in {U_r}^\star$ are such that $b - a \in \bar{F}_r$. Then $(b - a)^- + f \preccurlyeq (b - a)^+ + g$ and hence $a + f \preccurlyeq b + g$ for some $f, g \in {U_r}^\star$.

Assume second that $a \in \partial S_r$ and $b \in \sigma_{S_r}(a)$. Then $a \in \partial S$ and $b \in \sigma_S(a)$ and consequently $a \approx b$.

"If" part: Assume first that, for some $a, b \in {U_r}^\star$ and $f, g \in {U_a}^\star$, we have $(b+g)-(a+f) = (b-a)+(g-f) \in F$. Then $(b-a)^- + f \preccurlyeq (b-a)^+ + g$. Since $b - a \in \mathcal{D}(S_r)$, it follows $b - a \in \bar{F}_r$ and hence $a \mathrel{\bar{\preccurlyeq}_r} b$.

Assume second that $a + f \in \partial S$ and $b + g \in \sigma_S(a + f)$. If $f \neq 0$, we have $b = a$. If $f = 0$, we either have again $b = a$ or else $a \in \partial S_r$ and $b \in \sigma_{S_r}(a)$. We conclude that $a \mathrel{\bar{\approx}_r} b$. □

An immediate consequence of Lemma 5.6 is:

Lemma 5.7. $\preccurlyeq^r$ *is contained in* $\bar{\preccurlyeq}_r$.

The following proposition shows the meaning of the preorder $\bar{\preccurlyeq}_r$ in terms of the represented tomonoids.

Proposition 5.8. *The tomonoid* $(\langle {U_r}^\star \rangle_{\bar{\preccurlyeq}_r}; \bar{\preccurlyeq}_r, +, \{\bar{0}\})$ *is isomorphic with the quotient of* $(\langle \mathbb{N}^n \rangle_{\preccurlyeq}; \preccurlyeq, +, \{\bar{0}\})$ *by the ideal generated by* $\langle u \rangle_{\preccurlyeq}$, $u \in U_a$.

Proof. By Lemma 2.5, the quotient in question can be identified with the subtomonoid $\{\langle a + e \rangle_{\preccurlyeq} : a \in {U_r}^\star\}$ of $\langle \mathbb{N}^n \rangle_{\preccurlyeq}$, where e is a $\preccurlyeq$-maximal element of ${U_a}^\star$.

By Lemma 5.6, $a \mathrel{\bar{\preccurlyeq}_r} b$ iff $a + e \preccurlyeq b + e$, for any $a, b \in {U_r}^\star$. It follows that we can define the mapping

$$\langle {U_r}^\star \rangle_{\bar{\preccurlyeq}_r} \to \{ \langle a + e \rangle_{\preccurlyeq} \colon a \in {U_r}^\star \}, \quad \langle a \rangle_{\bar{\preccurlyeq}_r} \mapsto \langle a + e \rangle_{\preccurlyeq},$$

which is an isomorphism of tomonoids. □

We are ready to compile the characteristic properties of the direction cone (A, S, F) relative to its co-Archimedean subcone (A_r, S_r, F_r).

Proposition 5.9. *The following conditions hold:*

(E1) *Any $z \in F$ is of the form $z = y + f$, where $y \in \bar{F}_r$ and $f \in {U_a}^{\star\star}$.*

(E2) *Let $D \subseteq {U_r}^\star$ be a $\bar{\preccurlyeq}_r$-class and let $T_D = \{a + f \colon a \in D \text{ and } f \in {U_a}^\star\}$. If $c \in T_D \cap \mathring{S}$ and $d \in T_D \setminus \mathring{S}$, then $c - d \notin F$.*

(E3) *Let $D \subseteq {U_r}^\star$ be a $\bar{\preccurlyeq}_r$-class different from $\{\bar{0}\}$ and let $a \in D \cap S$. Then there is a $u \overset{\min}{\trianglelefteq} a$ such that either $a - u \in \mathring{S}$ or $a - u \notin D$.*

(E4) *Let $a \in \partial S$ such that $s(a) = 1$. Then there is a $v \overset{\min}{\trianglelefteq} a$ such that $a - v \in \mathring{S}$.*

(E5) *Let $y \in {U_r}^\star$ be such that $y^- \mathrel{\bar{\preccurlyeq}_r} y^+$, let $f \in {U_a}^{\star\star}$, and assume that $(y + f)^-, (y + f)^+ \in \partial S$. Then $y + f \in F$.*

Proof. Ad (E1): Let $y \in {U_r}^{\star\star}$ and $f \in {U_a}^{\star\star}$ and assume that $z = y + f \in F$. Then $y \in \mathcal{D}(S_r)$ and $y^- + f^- \preccurlyeq y^+ + f^+$, hence $y \in \bar{F}_r$.

Ad (E2): We first show the following:

($\star$) Let $a, b \in D$ and $f, g \in {U_a}^\star$ such that $a + f,\ b + g \notin \mathring{S}$. Then $a + f \approx b + g$.

Indeed, since $a + f$ and $b + g$ belong to an infinite $\preccurlyeq$-class, we have $a + f \approx a + e$ and $b + g \approx b + e$, where e is a $\preccurlyeq$-maximal element of ${U_a}^\star$. Furthermore, by Lemma 5.6, $a \mathrel{\bar{\approx}_r} b$ implies $a + e \approx b + e$, and ($\star$) follows.

Let now $c \in T_D \cap \mathring{S}$ and $d \in T_D \setminus \mathring{S}$. Since $c + e \notin \mathring{S}$ and hence c and $c + e$ are in different $\preccurlyeq$-classes, that is, $c \prec c + e$. Furthermore, by ($\star$), $c + e \approx d$. Hence $c \prec d$, and the claim is shown.

Ad (E3): Since $a \in S \setminus \{\bar{0}\}$, there is a $u \overset{\min}{\trianglelefteq} a$ such that $a - u \prec a$. Assume that $a - u \notin \mathring{S}$. We have $a - u \in S$, so this means $a - u, a \in \partial S$. We conclude that $(a - u) + f \approx a - u$ and $a + f \approx a$ for any $f \in {U_a}^\star$. It follows $a - u \mathrel{\bar{\prec}_r} a$ by Lemma 5.6, that is, $a - u \notin D$.

Ad (E4): Let a be as indicated. Because $a \in S \setminus \{\bar{0}\}$ and $s(a) = 1$, there is a $v \in U_a$ such that $a' = a - v \prec a$. Assume that $a' + u \notin S$ for some $u \in U_j$ such that $j \leqslant s(a')$. Then we would have $a' \approx a' + u$ and hence, because either v and u are Archimedean equivalent or $v \lll u$, $a' \approx a' + v = a$ as well. We conclude that $a' \in \mathring{S}$.

Ad (E5): By Lemma 5.6, we have $y^- + e \preccurlyeq y^+ + e$, where e is again a $\preccurlyeq$-maximal element of ${U_a}^\star$. Because $(y + f)^-$ and $(y + f)^+$ belong to infinite $\preccurlyeq$-classes, $(y +$

$f)^- = y^- + f^- \approx y^- + e$ and $(y+f)^+ = y^+ + f^+ \approx y^+ + e$. Thus $(y+f)^- \preccurlyeq (y+f)^+$, that is, $y + f \in F$. $\square$

6 Extension of direction f-cones

We have seen that a direction f-cone inducing a non-Archimedean preorder may be reduced to its co-Archimedean subcone; the number of Archimedean classes decreases in this way by one. The present section is devoted to the converse question: we wonder how we can extend a direction f-cone such that the co-Archimedean subcone of the new direction f-cone is the original one. To this end, we will show that the properties of direction f-cones listed in Proposition 5.9 are sufficient.

For the sake of consistency with the previous section, we will use the identical notation. However, our assumptions are different. We still assume that $A = (U_1, \ldots, U_m)$ is an ordered partition of $\mathcal{U}(\mathbb{N}^n)$, where $m \geqslant 2$. But this time, we assume that we are given the direction cone (A_r, S_r, F_r), inducing the monomial preorder $\preccurlyeq_r$ on ${U_r}^\star$. Our aim is to characterise those direction f-cones (A, S, F) whose co-Archimedean subcone is (A_r, S_r, F_r).

Theorem 6.1. *Let $F_r \subseteq \bar{F}_r \subseteq \mathcal{D}(S_r)$ be such that $(A_r, S_r, \bar{F}_r)$ fulfils* (Cf1)–(Cf4). *Let S be a finite, non-degenerate $\trianglelefteq$-ideal of $\mathbb{N}^n$ such that $S \cap {U_r}^\star = S_r$, let $F \subseteq \mathcal{D}(S)$ be such that $F_r = F \cap \mathcal{D}(S_r)$, and assume that (A, S, R) fulfils* (Cf1)–(Cf4). *Let furthermore* (E1)–(E5) *hold. Then (A, S, F) is a direction f-cone whose co-Archimedean subcone is (A_r, S_r, F_r).*

Moreover, all direction f-cones whose first component is A and whose co-Archimedean subcone is (A_r, S_r, F_r) are obtained in this way.

Proof. By Lemma 4.3, $(A_r, S_r, \bar{F}_r)$ induces a monomial preorder $\bar{\preccurlyeq}_r$ and (A, S, F) induces a monomial preorder $\preccurlyeq$. We prove some auxiliary facts.

(1) Let $a, b \in {U_r}^\star$ and $f, g \in {U_a}^\star$. If $a + f \preccurlyeq b + g$, then $a \mathrel{\bar{\preccurlyeq}_r} b$.

Indeed, assume first that $(b + g) - (a + f) \in F$. Then $b - a \in \bar{F}_r$ by (E1) and hence $a \mathrel{\bar{\preccurlyeq}_r} b$.

Assume second that $b+g \in \partial S$ and $a+f \in \sigma_S(b+g)$. Then either $a = b$, or otherwise $g = 0$ and $b \in \partial S_r$ and $a \in \sigma_{S_r}(b)$. Hence $a \mathrel{\bar{\approx}_r} b$. (1) follows.

(2) Let $D \subseteq S_r$ be a $\bar{\preccurlyeq}_r$-class, let $a \in T_D \cap \mathring{S}$ and $b \in T_D \setminus \mathring{S}$. Then $a \prec b$.

Indeed, $b \preccurlyeq a$ is, by (1), in contradiction to the following facts: $a - b \notin F$ by (E2); and rule (O2) cannot apply to a and b as $a \in \mathring{S}$.

(3) Let $a, b \in {U_r}^\star$. Then $a \mathrel{\bar{\preccurlyeq}_r} b$ if and only if $a + f \preccurlyeq b + g$ for some $f, g \in {U_a}^\star$.

The "if" part holds by (1).

To see the "only if" part, let $e \in U_a$ be such that $a \lhd e$ for any $a \in S \cap {U_a}^\star$. By (O2), we then have $e + u \approx e$ for any $u \in U_a$ and hence $e + f \approx e$ for any $f \in {U_a}^\star$. We conclude that $a + f \preccurlyeq b + g$ for some $f, g \in {U_a}^\star$ if and only if $a + e \preccurlyeq b + e$.

Assume now that $y = b - a \in \bar{F}_r$. If $-y \notin \bar{F}_r$, then $-y \notin F_r$ and consequently $-y \notin F$. Hence $y \in F$ and $a \preccurlyeq b$, and it follows $a + e \preccurlyeq b + e$. In the opposite case, we have $-y \in \bar{F}_r$, that is, $y^- \approx_r y^+$. We distinguish two cases.

Case 1. Let $y^- \in \mathring{S}$. We have $y^+ + e \notin S$, so in particular $y^+ + e \notin \mathring{S}$ and it follows $y^- \prec y^+ + e$ by (2). Hence $y^- + e \preccurlyeq y^+ + e + e \approx y^+ + e$ and we conclude $a + e \preccurlyeq b + e$.

Case 2. Let $y^- \in \partial S$. Let then $g \in U_a{}^\star$ be such that $y^+ + g \in \partial S$. We have $(y+g)^- = y^- \in \partial S$ and $(y+g)^+ = y^+ + g \in \partial S$. By (E5), $y + g \in F$. Hence $y^- \preccurlyeq y^+ + g$ and $y^- + e \preccurlyeq y^+ + g + e \approx y^+ + e$. It follows $a + e \preccurlyeq b + e$.

Assume second that $b \in \partial S_r$ and $a \in \sigma_{S_r}(b)$. Then $b \in \partial S$ and $a \in \sigma_S(b)$. Hence $a \approx b$ and $a + e \approx b + e$. The proof of (3) is complete.

Our next aim is to show that (A, S, F) is the direction cone of $\preccurlyeq$. By Lemma 4.4, $A_\preccurlyeq = A$.

We have to show $S_\preccurlyeq = S$. By Lemma 4.4, $S_\preccurlyeq \subseteq S$. To see the converse inclusion, let $a \in S$. We distinguish two cases.

Case 1. Let $a \in \mathring{S}$. Then, by (2) and (3), $\langle a \rangle_\preccurlyeq$ is a subset of $\mathring{S}$ and in particular finite. Hence $a \in \mathring{S}_\preccurlyeq \subseteq S_\preccurlyeq$.

Case 2. Let $a \in \partial S$. Let then $a = b + f$, where $b \in U_r{}^\star$ and $f \in U_a{}^\star$. Assume first that $f \neq 0$. Then, by (E4), $a - u \in \mathring{S}$ for some $u \overset{\min}{\trianglelefteq} a$. It follows $a - u \prec a$ by (2) and hence $a \in S_\preccurlyeq$. Assume second that $a = b \in S_r$. By (E3), there is a $u \overset{\min}{\trianglelefteq} a$ such that $a - u \in \mathring{S}$ or $a - u \mathrel{\bar{\prec}_r} a$. By (2) and (3), it follows $a - u \prec a$ and hence $a \in S_\preccurlyeq$.

It remains to show that $F = F_\preccurlyeq$. By Lemma 4.4, $F \subseteq F_\preccurlyeq$. To see the converse inclusion, let $z \in F_\preccurlyeq$. This means $z^- \preccurlyeq z^+$. If $z^- \in \mathring{S}$ or $z^+ \in \mathring{S}$, we have $z \in F$ by Lemma 4.4 again. Let $z^-, z^+ \in \partial S$. Let $y \in U_r{}^\star$ and $f \in U_a{}^\star$ be such that $z = y + f$. Then $y^- + f^- \preccurlyeq y^+ + f^+$ and, by (1), it follows that $y^- \mathrel{\bar{\preccurlyeq}_r} y^+$. By (E5), $z \in F$.

By construction, (A_r, S_r, F_r) is the co-Archimedean subcone of (A, S, F). Finally, the last statement is clear from Propositions 4.1, 5.5, and 5.9. □

7 Conclusion

Direction cones were introduced in [Vet2] as a means to describe finitely generated positive, commutative (p.c.) tomonoids. They can be characterised by three simple conditions. As they are infinite even if the represented tomonoid is finite, the question has remained open if there is a more appropriate approach to the finite case. In the present paper, we have discussed so-called direction f-cones, which can be considered as a finitary analogue of direction cones and are tailored solely to the description of finite p.c. tomonoids.

Direction f-cones are difficult to characterise. Starting from those properties that are found in analogy to the case of direction cones, we have seen, however, how finite p.c. tomonoids can be constructed. We have moreover specified the structure of a direction f-cone relatively to what we call its co-Archimedean subcone, which arises from "collapsing" the smallest Archimedean class. As a side effect, our results reveal to a good extent the structure of the congruences on $\mathbb{N}^n$ that lead to the monoidal reduct of a finite p.c. tomonoid.

Our work is intended to contribute to a better understanding of the algebras that are significant in fuzzy logic. In particular, finite MTL-chains are concerned. Accordingly, it would certainly be interesting to point out the significance of our results for unsolved questions related to many-valued logic. Furthermore, the relationship of our analysis to other approaches aiming at a classification of totally ordered monoids, like those that we mentioned in the introduction, could be worth an investigation. Finally, we have considered a quite special class of partially ordered monoids. It is open if our methods can be adapted to a broader class of pomonoids, for instance those that are not necessarily totally ordered, or those that are not necessarily positive.

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