Different representations of fuzzy vectors

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Dedicated to the memory of Dan Butnariu

Abstract. Fuzzy vectors were introduced as a description of imprecise quantities whose uncertainty originates from vagueness, not from a probabilistic model. Support functions are a classical tool for representation and computation with compact convex sets. The combination of these two techniques—support functions of fuzzy vectors—has been proposed by Puri and Ralescu. Independently, Bobylev proposed another type of support functions which allows a more economical representation. However, the form of the functions is not very intuitive. We suggest a new type of support functions which combines the advantages of both preceding approaches. We characterize the functions which are support functions of fuzzy vectors in the new sense.

1 Introduction

Fuzzy sets were suggested as a tool for computing with imprecise quantities. At each point of the n-dimensional real vector space $\mathbb{R}^n$, the membership function of a fuzzy set attains a real value from $[0, 1]$ describing to which extent this point is a satisfactory approximation of the desired value. In contrast to probability models, here we do not assume the existence of a random experiment deciding whether the point belongs (totally) to the set or not; belongness is a matter of degree expressed by the values from the unit interval.

In order to represent imprecise quantities, only some fuzzy subsets of $\mathbb{R}^n$ are adequate. A natural requirement is that they are *normal*, i.e., at least one point

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has the degree of membership 1 and can serve as a representative crisp value. Further, the membership function is expected to decrease with distance from this point. A slightly stronger requirement is that the level sets (cuts) are compact and convex. Under these assumptions, operations with fuzzy sets can be made by the Zadeh's extension principle, as well as by the Minkowski operations on level sets; both give the same results. Nevertheless, pointwise operations with sets are difficult to compute (even approximately). As an alternative, support functions were suggested as a tool representing fuzzy vectors. Operations with fuzzy vectors correspond to pointwise operations with support functions.

In this paper, we present and compare three types of support functions of fuzzy vectors, the first by Puri and Ralescu, the second by Bobylev, the third was introduced by Butnariu, Navara, and Vetterlein, but without any details on its properties. We fill this gap now. We give formulas for conversions between various types of support functions. We characterize those functions which can occur as support functions (of all the three types). The new type of support functions is often convex. However, we show that this is not a rule.

The paper is organized as follows: Section 2 summarizes known facts about support functions of crisp sets, Section 3 recalls the vertical and horizontal representations of fuzzy sets, with emphasis on fuzzy vectors. Section 4 describes particular types of support functions of fuzzy vectors, their characterizations and other properties. The final conclusions suggest possible applications of support functions in computing with fuzzy vectors.

2 Support functions of crisp sets

We denote by $\mathcal{K}^n$ the set of all non-empty compact convex subsets of $\mathbb{R}^n$. The set $\mathcal{K}^n$ is endowed with a linear structure (by $\mathbb{R}_+$ we denote the set of all non-negative reals):

$$\forall A, B \in \mathcal{K}^n : A + B = \{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{x} \in A,\ \boldsymbol{y} \in B\}\ ,$$
$$\forall A \in \mathcal{K}^n\ \forall \lambda \in \mathbb{R}_+ : \lambda\, A = \{\lambda\, \boldsymbol{x} \mid \boldsymbol{x} \in A\}\ .$$

For $r > 0$, we denote by $S^n_r \subset \mathbb{R}^n$ the sphere with diameter r, centered in the origin. If $r = 1$, we write $S^n = S^n_1$, and we denote by B^n the closed unit ball in $\mathbb{R}^n$. The open unit ball will be denoted by $B^n \setminus S^n$.

For any $A \in \mathcal{K}^n$, its *support function*, $h_A \colon \mathbb{R}^n \to \mathbb{R}$, is defined by

$$h_A(\boldsymbol{x}) = \max\{\langle \boldsymbol{p}, \boldsymbol{x}\rangle \mid \boldsymbol{p} \in A\}\,, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$ (see e.g. [10]). The mapping $A \mapsto h_A$ (defined on $\mathcal{K}^n$) is injective, i.e., each compact convex set is uniquely represented by its support function. Moreover, this mapping preserves the linear operations on $\mathcal{K}^n$ and ordering by inclusion:

$$\forall A, B \in \mathcal{K}^n : h_{A+B} = h_A + h_B\,, \tag{2}$$
$$\forall A \in \mathcal{K}^n\ \forall \lambda \in \mathbb{R}_+ : h_{\lambda\, A} = \lambda\, h_A\,, \tag{3}$$
$$\forall A, B \in \mathcal{K}^n,\ A \subseteq B : h_A \le h_B\,. \tag{4}$$

The functions which are support functions of compact convex sets can be characterized using the *epigraph* [8] (or *supergraph* [6]) of a function $h\colon \mathbb{R}^n \to \mathbb{R}$, i.e., the set

$$\mathrm{epi}_h = \left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 \geq h\,(x_1, \ldots, x_n)\right\}$$

of points which are above the graph of h.

Proposition 1. *A function $h\colon \mathbb{R}^n \to \mathbb{R}$ is a support function of some $A \in \mathcal{K}^n$ if and only if* epi_h *is a proper convex cone in $\mathbb{R}^{n+1}$, i.e.,* $\mathrm{epi}_h \neq \mathbb{R}^{n+1}$ *and*

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathrm{epi}_h \ \ \forall \lambda, \mu \in \mathbb{R}_+ : \lambda\, \boldsymbol{x} + \mu\, \boldsymbol{y} \in \mathrm{epi}_h\ .$$

In this case, $h = h_A$ for

$$A = \{\boldsymbol{p} \in \mathbb{R}^n \mid \forall \boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{p}, \boldsymbol{x} \rangle \leq h\,(\boldsymbol{x})\} \in \mathcal{K}^n\,. \tag{5}$$

For each $\boldsymbol{x} \in \mathbb{R}^n$, the set $\{\boldsymbol{p} \in \mathbb{R}^n \mid \langle \boldsymbol{p}, \boldsymbol{x} \rangle \leq h\,(\boldsymbol{x})\}$ is a closed halfspace; A is an intersection of such halfspaces, thus a closed convex set. Formula (1) (with sup instead of max) can be applied also in the more general case when A is an arbitrary bounded subset of $\mathbb{R}^n$; however, the result is the same as for the closed convex hull of A, thus the mapping is not injective when generalized to such sets. As the set epi_h is above a graph of a function which does not attain infinite values, $\mathrm{epi}_h - \mathrm{epi}_h = \mathbb{R}^n$. The support function is continuous.

Proposition 2. *The necessary and sufficient condition from Proposition 1 is equivalent to the conjunction of the following conditions:*

1. *Positive homogeneity:*
$$\forall \boldsymbol{x} \in \mathbb{R}^n \ \forall \lambda \in \mathbb{R}_+ : h\,(\lambda\, \boldsymbol{x}) = \lambda\, h\,(\boldsymbol{x})\ , \tag{6}$$
2. *Subadditivity:*
$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n : h\,(\boldsymbol{x} + \boldsymbol{y}) \leq h\,(\boldsymbol{x}) + h\,(\boldsymbol{y})\ , \tag{7}$$
3. *Continuity.*

As a consequence (for $\lambda = 0$ and $\boldsymbol{y} = -\boldsymbol{x}$), we obtain

$$\forall \boldsymbol{x} \in \mathbb{R}^n : h\,(\boldsymbol{x}) + h\,(-\boldsymbol{x}) \geq 0\,. \tag{8}$$

Due to (6), it is sufficient to know the values of h on a sphere S^n_λ for some $\lambda > 0$, i.e., the restricted support function $\eta_\lambda = h \restriction S^n_\lambda$. The reverse transformation is given by the formula

$$h\,(\boldsymbol{x}) = \frac{\|\boldsymbol{x}\|}{\lambda}\, \eta_\lambda \left(\frac{\lambda}{\|\boldsymbol{x}\|}\, \boldsymbol{x}\right)\,.$$

For a continuous function η_λ, condition (6) is useless and (7) attains the form

$$\begin{aligned}&\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\,,\ \boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{0}\ \forall \lambda \in \mathbb{R}_+ :\\ &\qquad \|\boldsymbol{x} + \boldsymbol{y}\|\, \eta_\lambda \left(\lambda\, \frac{\boldsymbol{x} + \boldsymbol{y}}{\|\boldsymbol{x} + \boldsymbol{y}\|}\right) \leq \|\boldsymbol{x}\|\, \eta_\lambda \left(\lambda\, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) + \|\boldsymbol{y}\|\, \eta_\lambda \left(\lambda\, \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right)\,.\end{aligned} \tag{9}$$

Following Bobylev [1], we call this property *quasiadditivity*.

Remark 1. In the sequel, we shall generalize the notion of support function to fuzzy vectors. In order to distinguish different terms, types of support functions will be specified by prefixes (e.g., PR-support function). When we want to emphasize that we speak of a support function of a *crisp* set, we speak of a *classical* support function.

3 Two representations of fuzzy sets and fuzzy vectors

Before extending the notion of support function to fuzzy sets, let us recall the properties of two representations of fuzzy sets (see, e.g., [4]) and their consequences for fuzzy vectors. In the sequel, they will be applied to descriptions of fuzzy vectors by support functions.

Let X be a non-empty crisp set (the *universe*) and V a fuzzy subset of X. The *vertical representation* of V is given by the *membership function* $m_V\colon X \to [0,1]$; $m_V(\boldsymbol{x})$ denotes the degree to which a point $\boldsymbol{x}$ belongs to V. The *horizontal representation* of V is given by the function $\ell_V\colon [0,1] \to 2^X$ which assigns to each $\alpha \in [0,1]$ the corresponding α-level set (α-cut)

$$\ell_V(\alpha) = [V]_\alpha = \{\boldsymbol{x} \in X \mid m_V(\boldsymbol{x}) \geq \alpha\} \subseteq X\,.$$

For $\alpha = 0$, the latter formula gives the whole space; instead of this, we make an exception (see [5, 7]) and define $\ell_V(0) = [V]_0$ as the closure of the set $\operatorname{supp}\boldsymbol{x} = \{\boldsymbol{x} \in \mathbb{R}^n \mid m_V(\boldsymbol{x}) > 0\} = \bigcup_{\alpha>0}[V]_\alpha$ ($\operatorname{supp}\boldsymbol{x}$ is called the *support* of $\boldsymbol{x}$).

The membership function can be any function $X \to [0,1]$. However, when the fuzzy set expresses an “imprecise quantity”, we often restrict attention to “meaningful” fuzzy subsets of $\mathbb{R}^n$; usual requirements are the following [4]:

1. *Normality:* $\exists \boldsymbol{x} \in \mathbb{R}^n : m_V(\boldsymbol{x}) = 1\,.$
2. *Fuzzy convexity:*
$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \ \forall \lambda \in [0,1] : m_V(\lambda\ \boldsymbol{x} + (1-\lambda)\ \boldsymbol{y}) \geq \min\{m_V(\boldsymbol{x}), m_V(\boldsymbol{y})\}\,.$$
3. *Boundedness:* $\operatorname{supp}\boldsymbol{x}$ is bounded.

A fuzzy set with these properties is called an (n-dimensional) *fuzzy vector* [5, 7]. We denote by $\mathcal{F}^n$ the set of all n-dimensional fuzzy vectors.

Remark 2. Normality says that there is at least one value $\boldsymbol{x}$ which belongs to V “completely” and can be considered a crisp representative value of the imprecise quantity V. Our definition (following [5, 7]) admits more such points. (In particular, crisp sets which are fuzzy vectors are not only singletons, but all non-empty compact convex sets.)

We have the following characterization of functions which appear in the horizontal representation [4]:

Proposition 3. *Let $\tau\colon\,]0,1] \to 2^X$ be a set-valued function. A necessary and sufficient condition for the existence of a (unique) fuzzy set V such that τ coincides with ℓ_V on $]0,1]$ is the conjunction of the following properties:*

1. *Monotonicity: τ is non-increasing, i.e.,*

$$\forall \alpha, \beta \in \left]0,1\right], \ \alpha \leq \beta : \tau(\alpha) \supseteq \tau(\beta) \ .$$

2. *Continuity: τ is left continuous, i.e., $\forall \beta \in \left]0,1\right] : \tau(\beta) = \bigcap_{\alpha<\beta} \tau(\alpha)$.*

In this case, $m_V(\boldsymbol{x}) = \sup\{\alpha \in \left]0,1\right] \mid \boldsymbol{x} \in \tau(\alpha)\}$ (where we put $\sup \emptyset = 0$).

Remark 3. If we want τ to be defined also at 0, then $\tau \colon [0,1] \to 2^X$ must be also right continuous at 0, i.e., $\tau(0) = \bigcup_{\alpha>0} \tau(\alpha)$. Due to left continuity, the value of τ at 1 is also unnecessary; the values of τ on $]0,1[$ suffice to determine the fuzzy set.

Fuzzy vectors in the horizontal representation can be characterized as follows:

Proposition 4. *Let $\tau \colon [0,1] \to 2^X$ be a set-valued function satisfying the conditions of Proposition 3 and Remark 3 and V be the corresponding fuzzy set V. Then V is a fuzzy vector if and only if all values of τ are non-empty compact convex sets, i.e., $\forall \alpha \in [0,1] : \tau(\alpha) \in \mathcal{K}^n$.*

4 Different types of support functions of fuzzy vectors

4.1 Approach by Puri and Ralescu

Following Puri and Ralescu [7] (see also the book by Diamond and Kloeden [5]), we define the *PR-support function* of a fuzzy vector $V \in \mathcal{F}^n$ as the function $H_V \colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ such that

$$H_V(\alpha, \boldsymbol{x}) = \sup\{\langle \boldsymbol{p}, \boldsymbol{x} \rangle \mid \boldsymbol{p} \in [V]_\alpha\}. \tag{10}$$

This means that, for each $\alpha \in [0,1]$, $H_V(\alpha, \cdot)$ is the (classical) support function of the crisp α-level set $[V]_\alpha$,

$$H_V(\alpha, \boldsymbol{x}) = h_{[V]_\alpha}(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. As each fuzzy vector is uniquely determined by its collection of level sets and these are described by their support functions, the PR-support function characterizes the fuzzy vector completely. (The 0-level set is not needed, because it does not carry any additional information.)

PR-support functions appeared useful in the computation of the Steiner point of a fuzzy vector [11] (as a useful reference point describing the position of the fuzzy vector).

Functions which can be PR-support functions of fuzzy vectors were characterized in [3]:

Theorem 1. *A function $\varphi\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is the PR-support function of a (unique) fuzzy vector $V \in \mathcal{F}^n$ if and only if it satisfies (the conjunction of) the following conditions:*

$$\forall \boldsymbol{z} \in \mathbb{R}^n \ \forall \lambda \in \mathbb{R}_+ : \varphi(\alpha, \lambda\, \boldsymbol{z}) = \lambda\, \varphi(\alpha, \boldsymbol{z})\,, \tag{11}$$

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n : \varphi(\alpha, \boldsymbol{x}+\boldsymbol{y}) \leq \varphi(\alpha, \boldsymbol{x}) + \varphi(\alpha, \boldsymbol{y})\,, \tag{12}$$

$$\varphi(\cdot, \boldsymbol{z}) \textit{ is non-increasing, left continuous on }]0,1]\,, \textit{ and right continuous at } 0\,. \tag{13}$$

The proof follows directly from Propositions 2 and 3.

Remark 4. As mentioned in Remark 3, the values for the first argument 0 or 1 are unnecessary for the representation. If they are omitted (only the restriction $\varphi \restriction]0,1[\times \mathbb{R}^n$ is used), Theorem 1 works if the requirement that $\varphi(\cdot, \boldsymbol{z})$ is right continuous at 0 is replaced by boundedness of φ.

As in the case of classical support functions, the representation of a fuzzy vector by the PR-support function $H_V\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is redundant; the domain can be restricted to $[0,1] \times B^n$, $[0,1] \times S^n$, or another appropriate set. The restriction $\varphi = H_V \restriction [0,1] \times B^n$ was used in [5, 3]. The only difference in Theorem 1 is that (11), (12) are applied only if all arguments fall in the restricted domain. In particular, (11) can be formulated as

$$\forall \boldsymbol{z} \in B^n \ \forall \lambda \in [0,1] : \varphi(\alpha, \lambda\, \boldsymbol{z}) = \lambda\, \varphi(\alpha, \boldsymbol{z})\,. \tag{14}$$

For the restriction $\varphi = H_V \restriction]0,1] \times S^n$, (11) is irrelevant and (12) has to be modified as in (9):

$$\forall \boldsymbol{x}, \boldsymbol{y} \in S^n : \|\boldsymbol{x}+\boldsymbol{y}\|\, \varphi\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{\|\boldsymbol{x}+\boldsymbol{y}\|}\right) \leq \|\boldsymbol{x}\|\, \varphi\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) + \|\boldsymbol{y}\|\, \varphi\left(\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right)\,. \tag{15}$$

4.2 Approach by Bobylev

The redundancy of the PR-support function inspired Bobylev to a more economical representation [1]. Each sphere $\{\alpha\} \times S^n$, $\alpha \in]0,1]$, is mapped onto S^n_α by the mapping $(\alpha, \boldsymbol{x}) \mapsto \alpha\, \boldsymbol{x}$. All these spheres fit into the unit ball B^n. This reduces the dimensionality and facilitates computations with fuzzy vectors via their support functions. It also saves space in (approximate) computer representations of fuzzy vectors.

The *B-support function* of a fuzzy vector $V \in \mathcal{F}^n$ is defined as the function $\bar{H}_V\colon B^n \to \mathbb{R}$,

$$\bar{H}_V(\boldsymbol{x}) = \max\{\langle \boldsymbol{p}, \boldsymbol{x}\rangle \mid \boldsymbol{p} \in [V]_{\|\boldsymbol{x}\|}\}. \tag{16}$$

Apparently, it determines a fuzzy vector V uniquely. The conversions between the two representations can be made by the following formulas:

$$\forall \boldsymbol{x} \in B^n : \bar{H}_V(\boldsymbol{x}) = H_V(\|\boldsymbol{x}\|, \boldsymbol{x})\,,$$

$$\forall \boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\} \ \forall \alpha \in]0,1] : H_V(\alpha, \boldsymbol{y}) = \frac{\|\boldsymbol{y}\|}{\alpha}\, \bar{H}_V\left(\frac{\alpha}{\|\boldsymbol{y}\|}\, \boldsymbol{y}\right)\,.$$

For each $\alpha \in]0,1]$, the mapping $\boldsymbol{x} \mapsto H_V(\alpha, \boldsymbol{x})$ coincides with $\bar{H}_V$ on the sphere S^n_α.

Functions which can be B-support functions of fuzzy vectors were characterized in [1] (we keep the original names of properties, although some of them might be debatable):

Theorem 2. *A function $\varphi\colon B^n \to \mathbb{R}$ is the B-support function of a (unique) fuzzy vector $V \in \mathcal{F}^n$ if and only if it satisfies the following conditions:*

1) Upper semicontinuity:

$$\forall \boldsymbol{x} \in B^n : \varphi(\boldsymbol{x}) = \limsup_{\boldsymbol{y} \to \boldsymbol{x}} \varphi(\boldsymbol{y})\,. \tag{B1}$$

2) Positive semihomogeneity:

$$\forall \boldsymbol{x} \in B^n \ \forall \lambda \in [0,1] : \lambda\, \varphi(\boldsymbol{x}) \le \varphi(\lambda\, \boldsymbol{x})\,. \tag{B2}$$

3) Quasiadditivity:

$$\begin{gathered} \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y} \ne 0 \ \forall \lambda \in [0,1] : \\ \|\boldsymbol{x}+\boldsymbol{y}\|\, \varphi\left(\lambda\, \frac{\boldsymbol{x}+\boldsymbol{y}}{\|\boldsymbol{x}+\boldsymbol{y}\|}\right) \le \|\boldsymbol{x}\|\, \varphi\left(\lambda\, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) + \|\boldsymbol{y}\|\, \varphi\left(\lambda\, \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}\right)\,. \end{gathered} \tag{B3}$$

4) "Normality":

$$\forall \boldsymbol{x} \in B^n : \varphi(\boldsymbol{x}) + \varphi(-\boldsymbol{x}) \ge 0\,. \tag{B4}$$

5) Boundedness:

$$\sup\left\{ \frac{|\varphi(\boldsymbol{x})|}{\|\boldsymbol{x}\|} \;\middle|\; \boldsymbol{x} \in B^n, \boldsymbol{x} \ne 0 \right\} < \infty\,. \tag{B5}$$

6)

$$\varphi(\mathbf{0}) = 0. \tag{B6}$$

In this case, the fuzzy vector V such that $\bar{H}_V = \varphi$ is given by

$$[V]_\alpha = \left\{ \boldsymbol{p} \in \mathbb{R}^n \mid \forall \boldsymbol{x} \in S^n_\alpha : \langle \boldsymbol{p}, \boldsymbol{x} \rangle \le \bar{H}_V(\boldsymbol{x}) \right\}$$

for all $\alpha \in]0,1]$ and its membership function is

$$m_V(\boldsymbol{p}) = \max\left\{ \alpha \in [0,1] \mid \forall \boldsymbol{x} \in S^n_\alpha : \langle \boldsymbol{p}, \boldsymbol{x} \rangle \le \bar{H}_V(\boldsymbol{x}) \right\}$$

Remark 5. Following [2], (B2) could be called more precisely $[0,1]$-*superhomogeneity*; (6) is $[0,1]$-homogeneity or $\mathbb{R}_+$-homogeneity.

Remark 6. [1] The values of the B-support function at $\boldsymbol{x}$ such that $\|\boldsymbol{x}\| \in \{0,1\}$ are unnecessary for the representation (cf. Remark 3). Condition (B6) is used only for more consistent results, it is not needed for the description of the fuzzy vector.

Remark 7. In this context, upper semicontinuity implies that

$$\forall \boldsymbol{x} \in B^n : \varphi(\boldsymbol{x}) = \lim_{\gamma \to 1-} \varphi(\gamma\, \boldsymbol{x})\,.$$

Together with quasiadditivity, it also ensures that the restriction of φ to the sphere S^n_α, $\alpha \in]0,1]$, is continuous.

4.3 New approach to support functions

One disadvantage of the Bobylev representation is that it is not very intuitive. In particular, the B-support functions are usually neither convex nor concave. Besides, the conditions characterizing B-support functions are not very elegant.

We propose an alternative. It was first announced in [3], without any details on its properties. Here we give a characterization of functions obtained as support functions in this sense (the conditions are simpler than for the Bobylev representation). Like the Bobylev approach, the support functions are again defined on the unit ball, but the mapping is different. Each sphere $\{\alpha\} \times S^n$, $\alpha \in]0,1[$, is mapped onto $S^n_{1-\alpha}$ by the mapping $(\alpha, \boldsymbol{x}) \mapsto (1-\alpha)\, \boldsymbol{x}$. As a result, we obtain usually convex functions. Moreover, convexity with (B3) and (B6) appears to be a sufficient condition for a function to be a support function of a fuzzy vector in this sense.

The *BNV-support function* of a fuzzy vector $V \in \mathcal{F}^n$ is defined as the function $\widehat{H}_V \colon B^n \to \mathbb{R}$,

$$\widehat{H}_V(\boldsymbol{x}) = \sup\{\langle \boldsymbol{p}, \boldsymbol{x}\rangle \mid \boldsymbol{p} \in [V]_{1-\|\boldsymbol{x}\|}\}\,. \tag{17}$$

Apparently, it determines a fuzzy vector V uniquely. For each $\alpha \in]0,1]$, the mapping $\boldsymbol{x} \mapsto H_V(1-\alpha, \boldsymbol{x})$ coincides with $\widehat{H}_V$ on the sphere S^n_α. The conversions between the representations can be made by the following formulas:

$$\begin{aligned}
&\forall \boldsymbol{x} \in B^n : \widehat{H}_V(\boldsymbol{x}) = H_V(1-\|\boldsymbol{x}\|, \boldsymbol{x})\,,\\
&\forall \boldsymbol{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}\;\; \forall \alpha \in]0,1[: H_V(\alpha, \boldsymbol{y}) = \frac{\|\boldsymbol{y}\|}{1-\alpha}\, \widehat{H}_V\left(\frac{1-\alpha}{\|\boldsymbol{y}\|}\, \boldsymbol{y}\right)\,,\\
&\forall \boldsymbol{x} \in B^n \setminus (S^n \cup \{\mathbf{0}\}) : \widehat{H}_V(\boldsymbol{x}) = \frac{\|\boldsymbol{x}\|}{1-\|\boldsymbol{x}\|}\, \bar{H}_V\left(\frac{1-\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\, \boldsymbol{x}\right)\,,\\
&\forall \boldsymbol{x} \in B^n \setminus (S^n \cup \{\mathbf{0}\}) : \bar{H}_V(\boldsymbol{x}) = \frac{1-\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\, \widehat{H}_V\left(\frac{\|\boldsymbol{x}\|}{1-\|\boldsymbol{x}\|}\, \boldsymbol{x}\right)\,.
\end{aligned}$$

Example 1. Let $V \in \mathcal{F}^1$ be the triangular fuzzy number given by its membership function

$$m_V(x) = \begin{cases} -1 + 2\,x & \text{if } x \in [-1/2, 0]\,,\\ 1 - x & \text{if } x \in]0,1]\,,\\ 0 & \text{otherwise.} \end{cases}$$

From this we derive the horizontal representation:

$$\ell_V(\alpha) = \left[\frac{\alpha-1}{2}, 1-\alpha\right]\,,$$

and its support functions:

$$H_V(\alpha, x) = \begin{cases} \frac{\alpha-1}{2}\, x & \text{if } x \le 0,\\ (1-\alpha)\, x & \text{if } x > 0, \end{cases}$$

$$\bar{H}_V(x) = \begin{cases} \frac{-x-1}{2}\, x & \text{if } x \in [-1,0]\,,\\ (1-x)\, x & \text{if } x \in]0,1]\,, \end{cases}$$

$$\widehat{H}_V(x) = \begin{cases} \frac{x^2}{2} & \text{if } x \in [-1,0]\,,\\ x^2 & \text{if } x \in]0,1]\,. \end{cases}$$

Notice that the BNV-support function is convex.

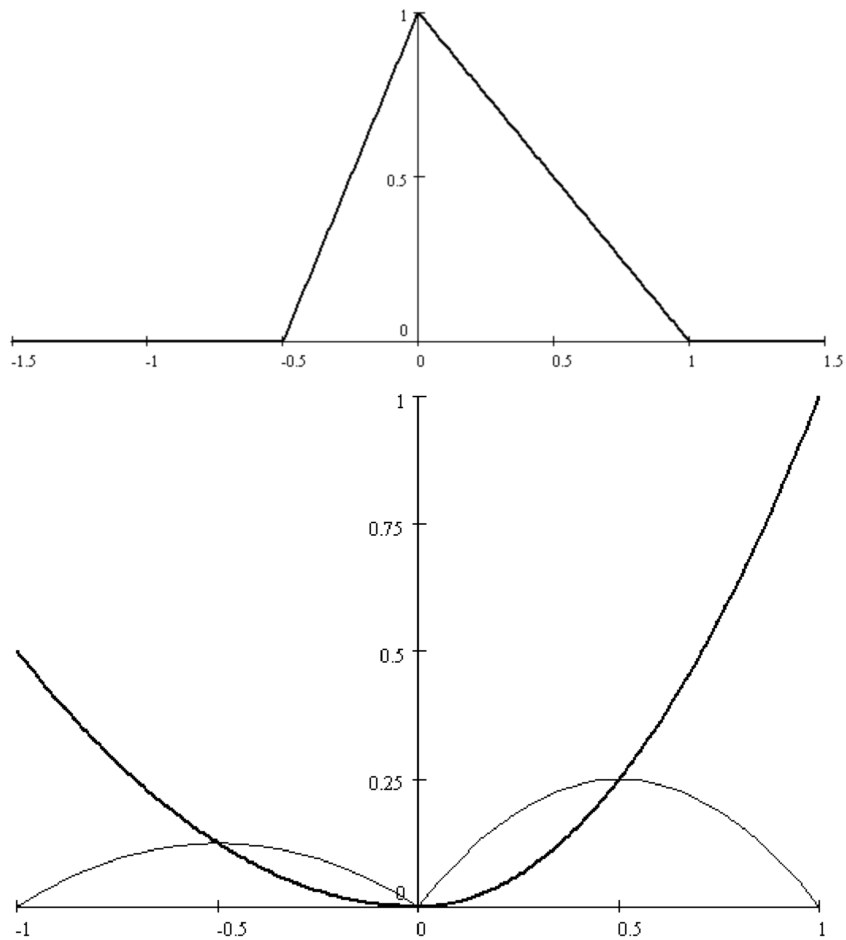

Fig. 1. The triangular fuzzy number of Example 1: membership function (top), B-support function (bottom thin), and BNV-support function (bottom thick).

BNV-support functions can be characterized as follows:

Theorem 3. *A function* $\varphi\colon B^n \to \mathbb{R}$ *is the BNV-support function of some fuzzy vector* $V \in \mathcal{F}^n$ *if and only if it satisfies conditions (B1), (B3), (B4) of Theorem 2 and*

$$\forall \boldsymbol{x} \in B^n\ \forall \lambda \in [0,1] : \lambda\,\varphi(\boldsymbol{x}) \geq \varphi(\lambda\,\boldsymbol{x})\,, \tag{B2'}$$

$$\varphi \textit{ is continuous at each point of } S^n\,. \tag{B5'}$$

In this case, the fuzzy vector V such that $\widehat{H}_V = \varphi$ is given by

$$[V]_\alpha = \left\{ \boldsymbol{p} \in \mathbb{R}^n \mid \forall \boldsymbol{x} \in S^n_{1-\alpha} : \langle \boldsymbol{p}, \boldsymbol{x} \rangle \leq \varphi(\boldsymbol{x}) \right\} \tag{18}$$

for all $\alpha \in \,]0,1]$, and its membership function is

$$m_V(\boldsymbol{p}) = \max \left\{ \alpha \in [0,1] \mid \forall \boldsymbol{x} \in S^n_{1-\alpha} : \langle \boldsymbol{p}, \boldsymbol{x} \rangle \leq \varphi(\boldsymbol{x}) \right\} \tag{19}$$

Remark 8. Following [2], (B2') could be called $[0,1]$-*subhomogeneity*.

Proof. We prove the sufficiency of the conditions.

In this context, upper semicontinuity implies that

$$\forall \boldsymbol{x} \in B^n \setminus S^n : \varphi(\boldsymbol{x}) = \lim_{\gamma \to 1+} \varphi(\gamma\, \boldsymbol{x}) \,. \tag{20}$$

Together with quasiadditivity, it also ensures that the restriction of φ to a sphere S^n_β, $\beta \in \,]0,1]$, is continuous (cf. Remark 7). For each $\alpha \in \,]0,1[$, upper semicontinuity and quasiadditivity imply that $\varphi \restriction S^n_{1-\alpha}$ is continuous; it can be extended to a positively homogeneous subadditive continuous function $\eta_\alpha \colon \mathbb{R}^n \to \mathbb{R}$ by the formula

$$\eta_\alpha(\boldsymbol{y}) = \begin{cases} \frac{||\boldsymbol{y}||}{1-\alpha}\, \varphi\left(\frac{1-\alpha}{||\boldsymbol{y}||}\, \boldsymbol{y} \right) & \text{if } \boldsymbol{y} \neq \boldsymbol{0}, \\ \boldsymbol{0} & \text{if } \boldsymbol{y} = \boldsymbol{0}. \end{cases}$$

It is a (classical) support function of some compact convex set, say $A_\alpha \subset \mathbb{R}^n$,

$$A_\alpha = \left\{ \boldsymbol{p} \in \mathbb{R}^n \mid \forall \boldsymbol{x} \in S^n_{1-\alpha} : \langle \boldsymbol{p}, \boldsymbol{x} \rangle \leq \varphi(\boldsymbol{x}) \right\} \in \mathcal{K}^n \,.$$

Let $0 < \alpha < \beta < 1$ and let A_α, A_β be the corresponding sets. For each $\boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, we apply (B2') to

$$\lambda = \frac{1-\beta}{1-\alpha} < 1 \,, \qquad \boldsymbol{x} = \frac{1-\alpha}{||\boldsymbol{y}||}\, \boldsymbol{y} \in S^n_{1-\alpha}$$

and obtain

$$\begin{aligned} \eta_\alpha(\boldsymbol{y}) &= \frac{||\boldsymbol{y}||}{1-\alpha}\, \varphi\left(\frac{1-\alpha}{||\boldsymbol{y}||}\, \boldsymbol{y} \right) = \frac{||\boldsymbol{y}||}{1-\alpha}\, \varphi(\boldsymbol{x}) \\ &\geq \frac{||\boldsymbol{y}||}{1-\alpha}\, \frac{1}{\lambda}\, \varphi(\lambda\, \boldsymbol{x}) = \frac{||\boldsymbol{y}||}{1-\beta}\, \varphi\left(\frac{1-\beta}{||\boldsymbol{y}||}\, \boldsymbol{y} \right) = \eta_\beta(\boldsymbol{y}) \,. \end{aligned}$$

Thus $\eta_\alpha \geq \eta_\beta$ and, by (4), $A_\alpha \supseteq A_\beta$. From (20) we infer that

$$\eta_\alpha = \lim_{\beta \to \alpha -} \eta_\beta$$

and hence

$$A_\alpha = \bigcap_{\beta < \alpha} A_\beta \,.$$

We proved that the mapping $\ell_V \colon \alpha \mapsto \ell_V(\alpha) = A_\alpha$ is a horizontal representation of a fuzzy set, in fact a fuzzy vector V whose level sets are of the form (18). The form (19) is obtained by the standard conversion to the vertical representation.

We omit the proof of the necessity of the conditions. It is easier and it follows the ideas used above.

Remark 9. The values of the BNV-support function at $\boldsymbol{x}$ such that $\|\boldsymbol{x}\| \in \{0,1\}$ are unnecessary for the representation (cf. Remark 3).

Remark 10. For B-support functions, (B5) guaranteed the boundedness of support of the corresponding fuzzy vector. BNV-support functions satisfy (B5) as a consequence of (B2'). On the other hand, boundedness is expressed by (B5'). By Remark 9, the values on S^n are unnecessary; as limits of values on smaller spheres, they do not bring any new information. Thus it is enough to define the BNV-support functions on the open unit ball $B^n \setminus S^n$. However, even then a boundedness condition is necessary. One possible formulation is that the BNV-support function is bounded on the open unit ball.

BNV-support functions satisfy also (B6). (This condition refers to the value at $\mathbf{0}$ which is unnecessary due to Remark 9.) Indeed, (B2') for $\lambda = 0$ implies $0 = 0\,\varphi(\boldsymbol{x}) \geq \varphi(0\,\boldsymbol{x}) = \varphi(\mathbf{0})$ and (B4) for $\boldsymbol{x} = \mathbf{0}$ gives $\varphi(\mathbf{0}) \geq 0$.

Proposition 5. *A convex function $\varphi\colon B^n \to \mathbb{R}$ satisfying (B3) and (B6) is the BNV-support function of some fuzzy vector.*

Proof. Convexity immediately implies (B2') and (B4). With quasiadditivity, we obtain boundedness, thus the function is also continuous. Theorem 3 applies and gives the desired fuzzy vector.

BNV-support functions are often convex (cf. Example 1). However, convexity is not a necessary condition in Proposition 5:

Example 2. In a 1-dimensional space $\mathbb{R}$, consider the fuzzy vector V with

$$m_V(p) = \sqrt{\max\{0, 1-|p|\}}\,.$$

Its horizontal representation is $\ell_V(\alpha) = \left[-\left(1-\alpha^2\right), 1-\alpha^2\right]$ and BNV-support function $\widehat{H}_V(x) = \left(1-(1-|x|)^2\right)|x|$. Its second derivative, $\widehat{H}''_V(x) = 4-6\,|x|$, is negative for $|x| > 2/3$.

5 Conclusions – computations with support functions

For all types of support functions introduced in this paper, the linear operations on $\mathcal{F}^n$ correspond to the pointwise operations on the support functions as in (2), (3). For the computation of Steiner points of fuzzy vectors (see [11]), all types of support functions are equally useful, but B- and BNV-support functions require one dimension less than PR-support functions.

Hausdorff metric on $\mathcal{K}^n$ gives rise to a metric on $\mathcal{F}^n$ which corresponds to the L_∞ (sup) norm on the space of PR-support functions on $[0,1] \times B^n$ (see [5] for details on this isometry). PR-support functions on $[0,1] \times S^n$ can be considered as well. Bobylev [1] introduced a metric on B-support functions on $[0,1] \times B^n$ and the isometry to the former metric was proved in [9]. Similar results can be obtained for BNV-support functions.

The advantage of BNV-support functions is that they form an irredundant representation in an n-dimensional domain (as well as B-support functions) while having a (usually convex) shape that reflects the properties of the fuzzy vector in a more intuitive way.

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