Left-continuous t-norms as functional algebras

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Abstract

With a left-continuous t-norm $\odot$, we may associate the set of its vertical cuts, namely, the set F of functions $f_a : [0,1] \to [0,1], \; x \mapsto x \odot a$. Endowed with the pointwise order, with the functional composition, with the constant 0 function and with the identity function, F is an algebra which is isomorphic to $([0,1]; \leq, \odot, 0, 1)$.

We characterize the functional algebras arising in this way from left-continuous t-norms; the key property is that every two functions commute. On the basis of this approach, we describe a subclass of the left-continuous t-norms in a unified way. This subclass comprises most left-continuous t-norms discussed in the literature.

Keywords: Left-continuous t-norms, functional algebra.

1 Introduction

When compiling the most basic properties which a function interpreting the conjunction in fuzzy logics should fulfill, one arrives at the notion of a left-continuous t-norm; see e.g. [7]. The logic which is based on all left-continuous t-norms together with their respective residua, is MTL [2].

It has been an open problem for many years how to describe the structure of left-continuous t-norms in general. A general approach being missing, a great deal of work has been done to define various ways to construct specific such two-placed functions. See [5] for an overview, and see also [10].

This paper is meant as a contribution to the difficult problem how to systematize left-continuous t-norms. We have developed a simple idea with the aim to bring order into the picture, which presently causes the impression that just a little bit of creativity is needed to find once again a new left-continuous t-norm, seemingly unrelated to earlier defined ones.

Still, what we offer is a way to construct left-continuous t-norms, and not all of them are included. However, our approach differs from previous ones. In the works of S. Jenei and others (see [5]), the geometrical viewpoint is frequently stressed, and the t-norm is visualized by its three-dimensional graph. In our own previous work [9, 10], the relationship between MTL-algebras and partially ordered groups is examined.

The present work stresses again the geometric viewpoint; but no three-dimensional objects are considered, we restrict to two dimensions. We simply work with the collection of vertical cuts through the three-dimensional graph. In this way, it is possible to visualize in a pleasant way our object of study. We are given a set of functions from the real unit interval into itself, and the basic property of this set is that every two functions commute. Indeed, the commutativity of the functional composition corresponds to both the associativity and the commutativity of the underlying t-norm.

When taking into account the typical examples of left-continuous t-norms, we see that the associated functional algebras show certain regularities. We have collected these regularities and derived from them a general way to construct left-continuous t-norms. In this way, we systematize a class of t-norms which have a particularly simple structure, but is not too restrictive either.

In this paper, we present our idea and its mathematical specification, together with a demonstration how the method works in known cases. The further elaboration is the subject of a forthcoming paper.

2 Preliminaries

The basic notion of this paper is the following one.

Definition 2.1 An operation $\odot\colon [0,1]^2 \to [0,1]$ is called a *t-norm* if, for all $a,b,c \in [0,1]$, (i) $(a \odot b) \odot c = a \odot (b \odot c)$, (ii) $a \odot b = b \odot a$, (iii) $a \odot 1 = a$, and (iv) $a \le b$ implies $a \odot c \le b \odot c$. A t-norm $\odot$ is called *left-continuous*, or *l.c.* for short, if for each $a \in [0,1]$, the function $(0,1] \to [0,1],\ \ x \mapsto x \odot a$ is left-continuous.

Let $\odot$ a l.c. t-norm. Then we call $([0,1]; \le, \odot, 0, 1)$ the *t-norm monoid* based on $\odot$.

Note that usually, we associate with a l.c. t-norm the MTL-algebra $([0,1]; \wedge, \vee, \odot, \to, 0, 1)$, where $\to$ is the residuum belonging to $\odot$. Classifying the t-norm algebras is, however, the same as to classify the t-norm monoids, as the operations $\odot$ and $\to$ are mutually definable. We will deal here only with the monoidal operation $\odot$.

The article is based on a simple observation, subject of the subsequent Theorem 2.3.

Definition 2.2 Let $\odot$ be a l.c. t-norm. For any $a \in [0,1]$, call

$$f_a^{\odot}\colon [0,1] \to [0,1],\ \ x \mapsto x \odot a$$

the *vertical cut*, or *cut* for short, of $\odot$ at the point a. Moreover, call

$$F^{\odot} = \{f_a^{\odot}\colon a \in [0,1]\}$$

cut set belonging to $\odot$.

We will denote the cut of a l.c. t-norm at a simply by f_a and the cut set by F, since the reference to a specific t-norm will always be clear.

Theorem 2.3 *Let $\odot$ be a l.c. t-norm. Then the cut set F belonging to $\odot$ is a set of functions from $[0,1]$ to $[0,1]$ with the following properties:*

(T1) *Every $f \in F$ is increasing.*

(T2) *Every two functions in F commute, that is, $f \circ g = g \circ f$ for any $f, g \in F$.*

(T3) *For every $a \in [0,1]$, there is exactly one $f \in F$ such that $f(1) = a$.*

(T4) *Every $f \in F$ is on $(0,1]$ left-continuous.*

Conversely, let F be a set of functions from $[0,1]$ to $[0,1]$ fulfilling (T1)–(T4)*. Then F is the cut set of a l.c. t-norm $\odot$, and $\odot$ is uniquely determined by*

$$a \odot b = f(a), \text{ where } f \in F \text{ is such that } f(1) = b. \quad (1)$$

Proof. Clearly, the cut set of a l.c. t-norm fulfills the conditions (T1)–(T4).

Conversely, let F be a set of functions from $[0,1]$ to itself such that (T1)–(T4) hold. Using (T3), denote by f_a, where $a \in [0,1]$, the unique element of F such that $f_a(1) = a$. We may then define $\odot$ by (1).

From (T2), we conclude $a \odot b = f_b(a) = f_b(f_a(1)) = f_a(f_b(1)) = f_a(b) = b \odot a$. Again by (T2), we have $(a \odot b) \odot c = f_c(a \odot b) = f_c(f_b(a)) = f_b(f_c(a)) = (a \odot c) \odot b$, and since $\odot$ is proved to be commutative, $\odot$ is associative. Moreover, $a \odot 1 = 1 \odot a = f_a(1) = a$. Finally, $a \le b$ implies $a \odot c = f_c(a) \le f_c(b) = b \odot c$ by (T1). So $\odot$ is a t-norm, which is left-continuous by (T4).

Obviously, F is the cut set belonging to $\odot$. In particular, f_a is the cut of $\odot$ at a. □

We shall endow the cut set belonging to a t-norm with an algebraic structure. For pairs of functions $f, g\colon [0,1] \to [0,1]$, we denote the pointwise order by $\le$. Furthermore, with each pair $f, g\colon [0,1] \to [0,1]$, we associate their composition $f \circ g$. The basic properties of the order $\le$ and of the operation $\circ$ on F are as follows.

Lemma 2.4 *Let F be a cut set belonging to some l.c. t-norm. Then:*

(T5) *The pointwise defined partial order $\le$ on F is a total order which is moreover dense, separable, complete, and bounded. The lower bound is*

$$\bar{0}\colon [0,1] \to [0,1],\ \ x \mapsto 0,$$

and the upper bound is

$$id\colon [0,1] \to [0,1],\ \ x \mapsto x.$$

Moreover, all suprema are calculated pointwise.

(T6) *F is closed under functional composition, that is, $f \circ g \in F$ for any $f, g \in F$.*

We note that, by Theorem 2.3, the properties (T5)–(T6) are consequences of (T1)–(T4).

We summarize what we have shown.

Definition 2.5 Let F be a cut set belonging to a l.c. t-norm $\odot$. Endow F with the pointwise order $\le$, with the composition of functions $\circ$, and with the constants $\bar{0}$ and id. Then we call $(F; \le, \circ, \bar{0}, id)$ the functional algebra belonging to $\odot$.

Theorem 2.6 *A l.c. t-norm algebra* $([0,1]; \le, \odot, 0, 1)$ *and the functional algebra* $(F; \le, \circ, \bar{0}, id)$ *belonging to*

$\odot$ are isomorphic. The isomorphism is given by $\Phi(a) = f_a$, where $a \in [0,1]$ and f_a is the unique element of F such that $f_a(1) = a$.

In this way, a one-to-one correspondence is defined between the l.c. t-norms and the algebras of functions from $[0,1]$ to $[0,1]$ fulfilling the properties (T1)–(T4).

Note that this correspondence can be extended to all t-norms, by dropping the condition (T4).

3 Examples

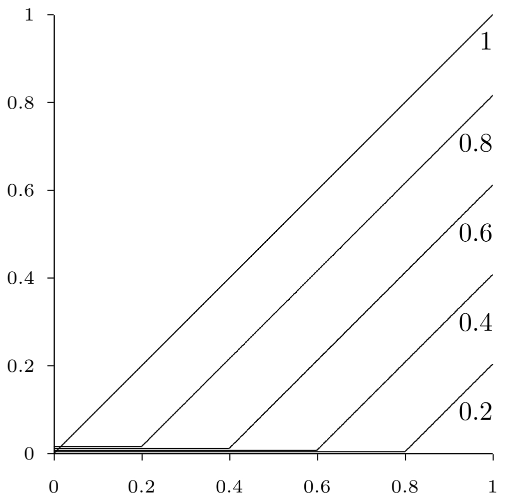
Let us have a look at the cut sets of some frequently encountered l.c. t-norms.

We consider first the continuous t-norms. Let $\odot_L$ be the Łukasiewicz t-norm: let

$$a \odot_L b = (a + b - 1) \vee 0$$

for $a, b \in [0,1]$. We plot the cuts of $\odot_L$ at the points $0.2, 0.4, ..., 1$:

The Łukasiewicz t-norm



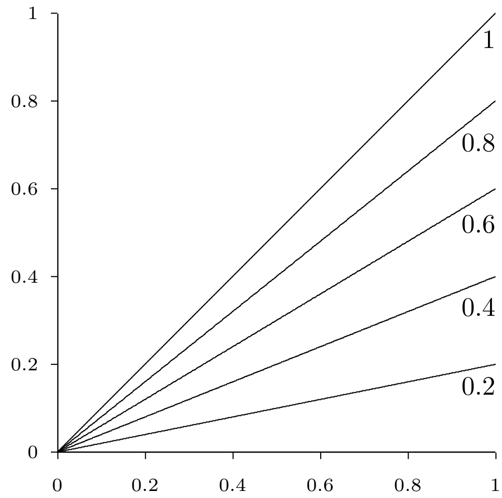
Note that our figures have only a schematic character; the cuts are drawn inaccurate in case that two of them overlap.

Next, let $\odot_P$ be the product t-norm: let

$$a \odot_P b = a \cdot b$$

for $a, b \in [0,1]$, where $\cdot$ denotes the multiplication of reals.

The product t-norm

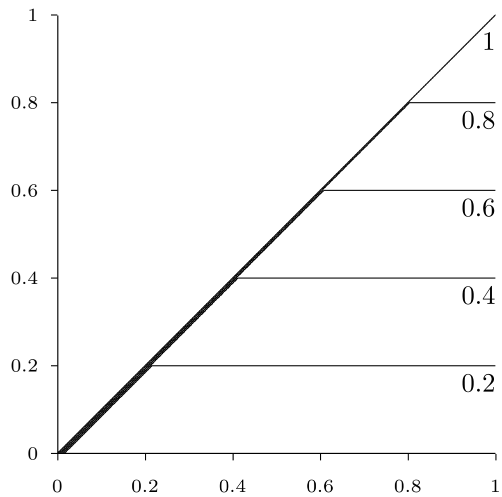


Finally, let $\odot_G$ be the Gödel t-norm: let

$$a \odot_G b = a \wedge b$$

for $a, b \in [0,1]$.

The Gödel t-norm



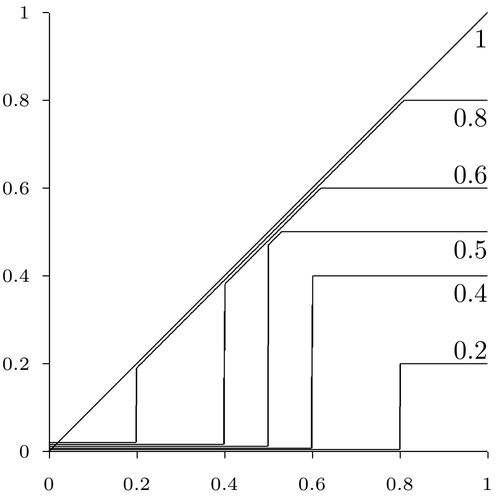
Any continuous t-norm arises from the three mentioned ones by means of an ordinal sum. So with these three examples, it is easy to see how the cut set of an arbitrary continuous t-norm looks like.

We now turn to examples of non-continuous l.c. t-norms. These are taken from the survey article [5], where further explanations and also the appropriate references can be found. Let $\odot_n$ be the nilpotent minimum t-norm: let

$$a \odot_n b = \begin{cases} a \wedge b & \text{if } a + b > 1, \\ 0 & \text{else} \end{cases}$$

for $a, b \in [0, 1]$. Discontinuity points are marked in the subsequent figures by vertical lines inserted into the graph, whose lower edge indicates the respective function value.

The nilpotent minimum t-norm

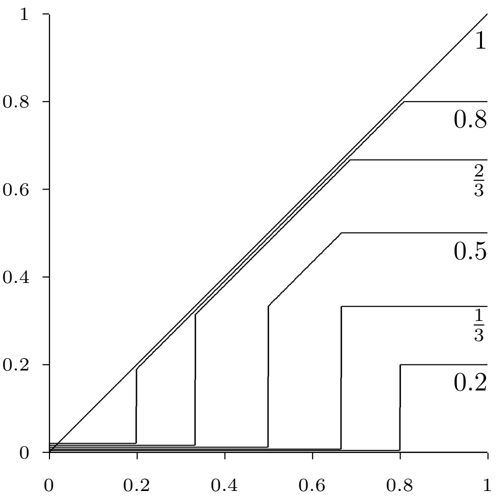


Next, we consider the t-norm $\odot_J$: let

$$a \odot_J b = \begin{cases} a \wedge b & \text{if } a+b>1 \text{ and } a \leq \frac{1}{3} \text{ or } a > \frac{2}{3}, \\ a+b-\frac{2}{3} & \text{if } a+b>1 \text{ and } \frac{1}{3} < a, b \leq \frac{2}{3}, \\ 0 & \text{if } a+b \leq 1 \end{cases}$$

for $a, b \in [0, 1]$.

The t-norm J

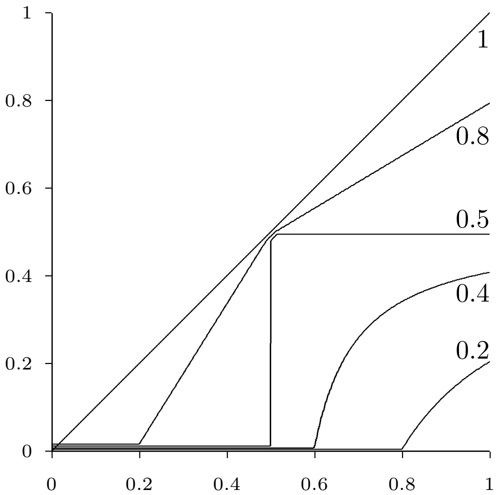


Furthermore, let $\odot_{rP}$ the rotated product t-norm: let

$$a \odot_{rP} b = \begin{cases} 2ab-a-b+1 & \text{if } a, b > \frac{1}{2}, \\ \frac{a+b-1}{2a-1} & \text{if } a > \frac{1}{2},\ b \leq \frac{1}{2}, \\ & \text{and } a+b>1, \\ 0 & \text{if } a+b \leq 1. \end{cases}$$

for $a, b \in [0, 1]$.

The rotated product t-norm

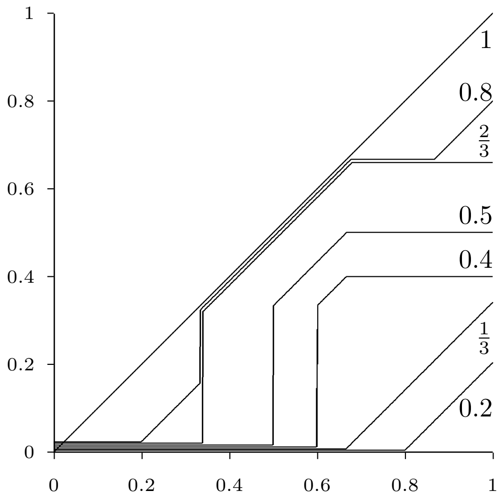


Finally, let $\odot_{raLL}$ be the rotation-annihilation of two Łukasiewicz t-norms: let

$$a \odot_{raLL} b = \begin{cases} a+b-1 & \text{if } a, b > \frac{2}{3} \text{ and } a+b > \frac{5}{3} \text{ or} \\ & a \leq \frac{1}{3},\ b > \frac{2}{3} \text{ and } a+b>1, \\ \frac{2}{3} & \text{if } a, b > \frac{2}{3} \text{ and } a+b \leq \frac{5}{3}, \\ a+b-\frac{2}{3} & \text{if } \frac{1}{3} < a, b \leq \frac{2}{3} \text{ and } a+b>1, \\ a & \text{if } \frac{1}{3} < a \leq \frac{2}{3},\ b > \frac{2}{3}, \\ 0 & \text{if } a+b \leq 1. \end{cases}$$

for $a, b \in [0, 1]$.

The rotation-annihilation of
two Łukasiewicz t-norms



4 The regular left-continuous t-norms

We will single out a class of left-continuous t-norms which allow a particularly easy description. All examples of the previous section belong to this class.

As the first step, we restrict the number of discontinuity points. Let $\odot$ be a l.c. t-norm; we introduce the following condition.

(R1) There is a an $n < \omega$ such that each cut of $\odot$ has at most n points of discontinuity.

From the universal-algebraic point of view, the class of l.c. t-norms fulfilling only this condition (R1) can be considered quite large. Namely, the algebras $([0,1]; \wedge, \vee, \odot, \rightarrow, 0, 1)$, where $\odot$ is a l.c. t-norm fulfilling (R1), generate the variety of MTL-algebras. Indeed, this variety is generated by its finite members [1], and every finite member can be isomorphically embedded into a t-norm algebra as explained in [6], and this t-norm algebra fulfills (R1).

Among the l.c. t-norm ruled out by (R1), we may mention the t-norm defined by Hájek in [4] and the t-norm defined by Hliněná in [8].

Our second condition refers directly to the representation of a l.c. t-norm $\odot$ by the corresponding cut set F. Recall that the cut of $\odot$ at 1 is the identity, the largest function in F; the smaller $a \in [0,1]$ gets, the smaller is the cut at a; finally, the cut at 0 is the zero function, the smallest function in F. In particular, the set of points which belong both to the graph of the identity function and to the closure of the graph of an $f_a \in F$ gets smaller if we make a smaller. It is this dependence which we assume to be of an easy kind.

Recall that for a left-continuous function $f\colon [0,1] \to [0,1]$, the right-side limit of f at a point x is denoted by $f^+(x)$.

Definition 4.1 For each $f \in F$, let

$$Q(f) = \{x \in [0,1] \colon f^+(x) = x\},$$

and put $q(a) = Q(f_a)$ for each $a \in [0,1]$. Moreover, let $\mathcal{C}([0,1])$ be the set of closed subsets of $[0,1]$ containing 0, and partially order $\mathcal{C}([0,1])$ by the set-theoretic inclusion.

Note that q is a monotonously increasing function from $[0,1]$ to $\mathcal{C}([0,1])$. The dependence of $q(a)$ from a is in the cases of commonly known l.c. t-norm simple, but can in general be arbitrarily complicated. This motivates us to introduce the following further condition on $\odot$. The power set of a set X will be denoted by $\mathcal{P}(X)$.

(R2) There are $0 = x_0 < x_1 < \ldots < x_k = 1$, $k \geq 1$, and $0 = a_0 < a_1 < \ldots < a_l = 1$, $l \geq 1$, such that, for every $1 \leq i < k$ and $1 \leq j < l$, the function

$$\begin{aligned} q_i^j\colon \quad [a_{j-1}, a_j) &\rightarrow \mathcal{P}((x_{i-1}, x_i)), \\ b &\mapsto q(b) \cap (x_{i-1}, x_i) \end{aligned}$$

is either constant $\emptyset$, or constant (x_{i-1}, x_i), or of the form $q(b) = (x_{i-1}, r(b))$ for some order-preserving bijection $r\colon [a_{j-1}, a_j) \to [x_{i-1}, x_i)$, or of the form $q(b) = (l(b), x_i)$ for some order-reversing bijection $l\colon [a_{j-1}, a_j) \to (x_{i-1}, x_i]$.

The condition (R2) is more restrictive than (R1). Let us call a t-norm *regular* if it fulfills (R1) and (R2). We have:

- All continuous t-norms are regular.
- A finite ordinal sum of regular l.c. t-norms is again regular.
- The t-norms constructed from regular l.c. t-norms by means of annihilation, rotation, or rotation-annihilation, as explained in [5], are again regular.

On the other side, an ordinal sum of countably many regular l.c. t-norms such that the number of discontinuity points of the cuts is not globally bounded, does not fulfill (R1). Furthermore, let M be the finite totally ordered monoid given in [3, Chapter 5] as an example for the property of not being formally integral. The dual of M expands to a finite MTL-algebra which, embedded into a t-norm algebra according to [6], results in a l.c. t-norm fulfilling (R1), but not (R2).

5 The description of regular left-continuous t-norms

Regular t-norms are made up from only six different kinds of “building blocks”.

The six possible constituents of regular l.c. t-norms can be identified with certain algebras of functions from $[0,1]$ to $[0,1]$.

Definition 5.1 Let C be a set of functions $f_a : [0,1] \to [0,1]$, where $0 \le a \le 1$, such that $f_0 = \bar{0}$, $f_1 = id$, and $f_0 \le f \le f_1$ for every $f \in C$. Endow C with the order $\le$, the functional composition $\circ$, and the constant id.

(i) Let $f_a \colon [0,1] \to [0,1]$, $x \mapsto a \cdot x$. Then C is called the *product algebra.*

(ii) Let $f_a \colon [0,1] \to [0,1]$, $x \mapsto (x+a-1) \vee 0$. Then C is called the *Łukasiewicz algebra.*

(iii) Let $f_a \colon [0,1] \to [0,1]$, $\frac{(a+x-1)\vee 0}{a}$ if $a > 0$. Then C is called the *reversed product algebra.*

(iv) Let $f_a \colon [0,1] \to [0,1]$, $x \mapsto x^{\frac{1}{a}}$ if $a > 0$. Then C is called the *power algebra.*

(v) Let

$$f_a(x) = \begin{cases} x & \text{for } x > a, \\ 0 & \text{for } x \le a \end{cases}$$

for $x \in [0,1]$. Then C is called the *left-idempotency algebra.*

(vi) Let

$$f_a(x) = \begin{cases} x & \text{for } x \le a, \\ a & \text{for } x \ge a \end{cases}$$

for $x \in [0,1]$. Then C is called the *right-idempotency algebra.*

In the sequel, we will actually have to do with sets of functions from an interval $(a,b]$, rather than $[0,1]$, to itself. Each of the algebras just defined may be transformed by means of an order-preserving bijection from $[0,1]$ to $(a,b]$, and will then be called a isomorphic to the algebra of the respective type.

Definition 5.2 Let $\odot$ be a l.c. t-norm and $(F; \le, \circ, \bar{0}, id)$ the functional algebra belonging to $\odot$, and let $0 \le a < b \le 1$. Let $F_{(a,b]}$ be the set of functions

$$f_{(a,b]} \colon [a,b] \to [a,b], \quad t \mapsto \begin{cases} f(t) & \text{if } t > a \text{ and } f(t) > a, \\ a & \text{else,} \end{cases}$$

where $f \in F$; and endow $F_{(a,b]}$ with the pointwise order $\le$, the functional composition $\circ$, and the identity $id|_{[a,b]}$. Then $(F_{(a,b]}; \le, \circ, id|_{[a,b]})$ is called the *basic algebra* belonging to the interval $(a,b]$.

Moreover, the *parameter set* belonging to $(a,b]$ is the smallest left-open right-closed interval $(u,v]$ such that, for any $f \in F$, $a < f(b) < b$ if and only if $u < f(1) < v$.

So given an interval $(a,b] \subseteq [0,1]$, the basic algebra belonging to $(a,b]$ arises from F by considering the graph of each cut f only inside the triangle with the points (a,a), (b,b), and (b,a). More precisely, we first restrict f to $(a,b]$; we map each $x \in (a,b]$ to $f(x)$ only in case that this value is larger than a, else we map x to a; and we map the remaining boundary point a to a.

The crucial observation is the following.

Theorem 5.3 *Let $\odot$ be a regular l.c. t-norm. Then there are $0 = a_0 < \ldots < a_k = 1$ such that, for each $i = 1, \ldots, k$, the algebra $F_{(a_{i-1},a_i]}$ is isomorphic to the product, Łukasiewicz, reversed product, power, left-idempotency, or right-idempotency algebra.*

According to this theorem, let us associate with a regular l.c. t-norm its *characteristic data*: (i) the *basic intervals* $(a_0,a_1], \ldots, (a_{k-1},a_k]$, (ii) for each $i = 1, \ldots, k$, the *type* of $F_{(a_{i-1},i]}$, which is one of product, Łukasiewicz, reversed product, power, left-idempotency, or right-idempotency, and (iii) for each $i = 1, \ldots, k$, the parameter set belonging to $(a_{i-1},a_i]$. We remark that this data can in general not be chosen unambiguously.

As an example, consider the t-norm $\odot_J$ above. In this case, the basic intervals are $(0,\frac{1}{3}]$, $(\frac{1}{3},\frac{2}{3}]$, and $(\frac{2}{3},1]$. $F_{(0,\frac{1}{3}]}$ is of type left-idempotency, $F_{(\frac{1}{3},\frac{2}{3}]}$ is of type Łukasiewicz, and $F_{(\frac{2}{3},1]}$ is of type right-idempotency. The parameter set belonging to $(0,\frac{1}{3}]$ and $(\frac{2}{3},1]$ is $(\frac{2}{3},1]$; the parameter set belonging to $(\frac{1}{3},\frac{2}{3}]$ is $(\frac{1}{3},\frac{2}{3}]$.

In this case, like in the case of all other examples given in Section 3, this information determines the t-norm, up to isomorphism, uniquely. However, this is not true in the general case.

Definition 5.4 We call a regular l.c. t-norm $\odot$ *locally determined* if any other regular l.c. t-norm with the same characteristic data is isomorphic to $\odot$.

One sufficient condition to ensure that a regular l.c. t-norm is locally determined is as follows. Here, a function f is called idempotent if $f \circ f = f$.

Proposition 5.5 *Let $\odot$ be a regular l.c. t-norm. Let the following condition hold:*

(R3) For each basic interval $(a,b]$ whose parameter is also $(a, b]$, the cut f_a restricted to $[0, a]$ is idempotent.

Then $\odot$ is locally determined.

6 Conclusion

We have presented an approach to describe the so-called regular l.c. t-norms in a unified way. The main tool is the algebra of commuting functions naturally associated with each t-norm. Not every l.c. t-norm is regular, but as measured by the set of l.c. t-norms which can be found in the literature, the class of regular t-norms is quite comprehensive.

Our method provides also a recipe how to construct new l.c. t-norms. To this end, the characteristic data is to be chosen tentatively in a way that, e.g. using condition (R3), a whole functional algebra is determined by it. It is then straightforward to check whether this functional algebra actually corresponds to a l.c. t-norm.

To characterize the exact rules according to which we have to associate with a number of basic intervals types and paramater sets, so as to get the characteristic data of a left-continuous t-norm, remains as a task for subsequent papers.

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