THE STEINER CENTROID OF FUZZY SETS *

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ABSTRACT:
With a crisp compact convex subset A of the $\mathbb{R}^n$ $(n \geq 1)$, we may associate one of its elements in a canonical way: the Steiner centroid $s(A)$ of A. This function s from the space of compact convex subsets of the $\mathbb{R}^n$ to the $\mathbb{R}^n$, may be unambiguously characterized by three simple properties.

In this paper, we consider the more comprehensive space $\mathcal{F}^n$ of fuzzy sets over $\mathbb{R}^n$, and we study functions from $\mathcal{F}^n$ to $\mathbb{R}^n$ fulfilling three properties defined analogously to the crisp case. We show that there is more than one such function, and we give a precise description of all of them. Apparently, no canonical choice among them is possible.

Keywords: Steiner point, convex fuzzy set, fuzzy vector, defuzzification

1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space, where $n \geq 1$, and let $\mathcal{K}^n$ the space of all compact convex subsets of $\mathbb{R}^n$. Intuitively, the transition from $\mathbb{R}^n$ to $\mathcal{K}^n$ may be viewed as an intermediate step towards the fuzzification of $\mathbb{R}^n$. In contrast to the precise location described by a point of the $\mathbb{R}^n$, a convex set from $\mathcal{K}^n$ provides only a bound for every direction in space.

According to this point of view, a closed convex set represents some value together with information about its impreciseness. The question naturally arises if every set $A \in \mathcal{K}^n$ may be reasonably viewed as a set around some central element $s(A) \in A$. The mapping $s \mapsto s(A)$ could then be considered as a "defuzzification" function.

Let us collect the minimal requirements which such a function s: $\mathcal{K}^n \to \mathbb{R}^n$ should fulfil. Probably most important, the point $s(A)$ should not depend on where and how A is positioned in space, that is, s should be invariant under Euclidean isometries. Moreover, s should respect the structure which $\mathcal{K}^n$ inherits from $\mathbb{R}^n$, both the linear and the topological one. Indeed, $\mathcal{K}^n$ is endowed with the pointwise addition generalizing the addition on $\mathbb{R}^n$ in a natural way; and $\mathcal{K}^n$ is endowed with the topology induced by the Hausdorff metric, which coincides with those induced by the L^p-metrics. So s should be compatible with the addition, and s should be continuous.

These three properties are fulfilled by the function s which associates with each compact convex subset its Steiner centroid. It was an open question for many years if there is any other such function, and it turned out that this is not the case; the Steiner centroid is unambigously defined by the mentioned three conditions [5, 3].

Let us now consider the more general situation that our space not only contains sharply limited subsets of a real Euclidean space, but also sets with unsharp boundary. The fuzzy analogue of the space $\mathcal{K}^n$ is the space $\mathcal{F}^n$, the set of standard fuzzy sets in the sense of Diamond and Kloeden [2]. We wonder what kind of "defuzzification" function exists which maps from $\mathcal{F}^n$ to $\mathbb{R}^n$ and which fulfils analogous conditions to those characteristic for the Steiner centroid of sharp subsets.

Steiner centroids on the space $\mathcal{F}^n$ of fuzzy sets are supposed to respect the structure inherent in $\mathcal{F}^n$. Namely, the linear structure of $\mathcal{K}^n$ generalizes in a straightforward way to $\mathcal{F}^n$; accordingly, fuzzy vectors are added in the usual way (see e.g. [2]). For the topology, however, we have to make a choice; in this paper, we will take the L^2-metric on $\mathcal{K}^n$ (again cf. e.g. [2]). The conditions for functions S from $\mathcal{F}^n$ to $\mathbb{R}^n$ may then be chosen in complete analogy to the crisp case. Namely, S should be invariant under Euclidean isometries; S should preserve the addition; and S should be continuous w.r.t. the L^2-topology.

We succeeded to determine all functions fulfilling these three properties; this the main result of this paper. As our formulation already suggests, there is more than one; so uniqueness is lost. Moreover, as the representation of these functions will show, it is hardly possible to make a canonical choice.

2 Convex sets and the Steiner centroid

Let us fix some $n \geq 2$. By $\mathcal{K}^n$, we denote the set of compact convex subsets of $\mathbb{R}^n$. The (Minkowski) addition and multiplication by positive reals, are defined pointwise on $\mathcal{K}^n$. By S^{n-1} we denote the unit sphere in $\mathbb{R}^n$. For any $A \in \mathcal{K}^n$, we let

$$h_A\colon\ S^{n-1} \to \mathbb{R},\quad e \mapsto \max\{(a,e)\colon\ a \in A\}$$

be the *support function* of A (see e.g. [4]). Addition and multiplication by positive reals in $\mathcal{K}^n$ coincide with the same operations on the corresponding support functions.

We shall embed $\mathcal{K}^n$ into the space $L^2(S^{n-1})$ (see e.g. [2]). Namely, let $L^2(S^{n-1})^{\mathcal{K}}$ be the subset of $L^2(S^{n-1})$ containing the support functions of the elements of $\mathcal{K}^n$. Then $L^2(S^{n-1})^{\mathcal{K}}$ is a positive cone in $L^2(S^{n-1})$; the subspace $L^2(S^{n-1})^{\mathcal{K}} - L^2(S^{n-1})^{\mathcal{K}}$ of differences of pairs from $L^2(S^{n-1})^{\mathcal{K}}$ is dense in $L^2(S^{n-1})$. In the sequel, we will identify $\mathcal{K}^n$ with $L^2(S^{n-1})^{\mathcal{K}}$,

*The support of grant 201/02/1540 of the Grant Agency of the Czech Republic is gratefully acknowledged.

that is, we will treat $\mathcal{K}^n$ as a subset of $L^2(S^{n-1})^{\mathcal{K}}$. We thus in particular endow $\mathcal{K}^n$ with the $||\cdot||_2$-metric. This metric is equivalent to the Hausdorff metric of $\mathcal{K}^n$.

The investigations of this paper are based on the following facts [5, 3].

Definition 2.1 The *Steiner centroid* of $A \in \mathcal{K}^n$ is defined by

$$s(A) = \frac{1}{V(B^n)} \int_{S^{n-1}} h_A(e)\, e\, d\lambda(e),$$

where $e \in S^{n-1}$ varies over the unit vectors of $\mathbb{R}^n$, λ is the Lebesgue measure on S^{n-1}, and $V(B^n)$ is the volume of the unit ball B^n of $\mathbb{R}^n$.

Notice that $s(A) \in A$.

Theorem 2.2 *Let s': $\mathcal{K}^n \to \mathbb{R}^n$ have the following properties:*

(S1) *For any $A, B \in \mathcal{K}^n$, $s'(A+B) = s'(A) + s'(B)$.*

(S2) *For $A \in \mathcal{K}^n$ and any Euclidean isometry τ of $\mathbb{R}^n$, we have $s'(\tau A) = \tau s'(A)$.*

(S3) *s' is continuous.*

Then $s' = s$.

This theorem was proved first for the case $n = 2$ by Shephard [5] and later for the case $n \geq 3$ by Schneider [3].

3 Fuzzy sets and Steiner centroids

Let us denote by $\mathcal{F}^n$ the collection of all n-dimensional standard convex fuzzy sets in the sense of Diamond and Kloeden [2] (*fuzzy vectors* in the terminology of [1]), that is, the set of all mappings v: $\mathbb{R}^n \to [0,1]$ such that $v^{-1}(1) \neq \emptyset$, the closure of the support of v is bounded, v is upper semicontinuous, and v is fuzzy convex. In the usual way, we define addition and multiplication by positive reals on $\mathcal{F}^n$.

It is advantageous to view a fuzzy set v as a function from the levels $\alpha \in (0,1]$ to the corresponding level sets $[v]^\alpha = \{x \in \mathbb{R}^n : v(x) \geq \alpha\}$. Indeed, $\mathcal{F}^n$ may be identified with the bounded, nonincreasing, left-continuous functions from $(0,1]$ to $\mathcal{K}^n$. As $\mathcal{K}^n$ is in turn represented by continuous functions from S^{n-1} to $\mathbb{R}$, $\mathcal{F}^n$ corresponds to a set of continuous real-valued functions on $(0,1] \times S^{n-1}$. The addition and multiplication by positive reals in $\mathcal{F}^n$ coincides with the same operations performed pointwise with the corresponding functions.

We shall embed $\mathcal{F}^n$ into $L^2((0,1] \times S^{n-1})$. Let $L^2((0,1] \times S^{n-1})^{\mathcal{F}}$ be the set of functions which represent fuzzy sets; then $L^2((0,1] \times S^{n-1})^{\mathcal{F}}$ is a positive cone in $L^2((0,1] \times S^{n-1})$. Again, we will identify $\mathcal{F}^n$ and $L^2((0,1] \times S^{n-1})^{\mathcal{F}}$. In particular, $\mathcal{F}^n$ is endowed with the $||\cdot||_2$-metric (cf. also [2]). Moreover, we shall treat $\mathcal{K}^n$ as a subset of $\mathcal{F}^n$.

We are going to investigate the following functions.

Definition 3.1 Let us call a function S: $\mathcal{F}^n \to \mathbb{R}^n$ a *Steiner centroid* if S has the following properties:

(SF1) For any $v, w \in \mathcal{F}^n$, $S(v+w) = S(v) + S(w)$.

(SF2) For any Euclidean isometry τ of $\mathbb{R}^n$ and any $v \in \mathcal{F}^n$, we have $S(\tau v) = \tau S(v)$, where $\tau v = v \circ \tau^{-1}$ (v being seen as a function from $\mathbb{R}^n$ to $[0,1]$).

(SF3) S is continuous.

Let us first see how we may generate typical examples of Steiner centroids.

Proposition 3.2 *Let $\mu \in L^2((0,1])$ be a non-negative function such that $\int_{(0,1]} \mu(\alpha)\, d\alpha = 1$. Define for $v \in \mathcal{F}^n$*

$$S_\mu(v) = \int_{(0,1]} s([v]^\alpha)\, \mu(\alpha)\, d\alpha,$$

where s is the classical Steiner centroid of the (crisp) level set $[v]^\alpha$. Then S_μ is a Steiner centroid.

It follows in particular that by the properties (SF1)–(SF3), a Steiner centroid cannot be defined unambiguously. To impose further properties on S to obtain uniqueness is amazingly difficult; it is an open question if this is possible in some reasonable, well motivated way.

4 Characterization of Steiner centroids

We shall prove that any Steiner centroid is of the form S_μ for some $\mu \in L^2((0,1])$, that is, the example of Proposition 3.2 reflects the most general case.

Theorem 4.1 *Let S: $\mathcal{F}^n \to \mathbb{R}^n$ be a Steiner centroid. Then there is a non-negative function $\mu \in L^2((0,1])$ fulfilling $\int_{(0,1]} \mu(\alpha)\, d\alpha = 1$ such that $S = S_\mu$.*

Proof. Let us first note that the restriction of S to $\mathcal{K}^n$ fulfils (S1)–(S4) of Proposition 2.2, whence $S(A) = s(A)$ for all $A \in \mathcal{K}^n$.

Now, S is by assumption associated to a continuous positive linear mapping from the positive cone $L^2((0,1] \times S^{n-1})^{\mathcal{F}}$ to $\mathbb{R}^n$. It may be unambiguously extended to a linear mapping on the subspace of all differences of elements of this cone, and then extended to the whole space $L^2((0,1] \times S^{n-1})$; we still denote this mapping by S. It follows that we may write $S(v) = \int_{(0,1]} T_\alpha([v]^\alpha)\, d\alpha$ for certain linear mappings T_α: $L^2(S^{n-1}) \to \mathbb{R}^n$, $\alpha \in [0,1]$, such that $\int_{(0,1]} T_\alpha(A)\, d\alpha = s(A)$ for all $A \in \mathcal{K}^n$ (we again identify the level sets $[v]^\alpha$ with their support functions).

We shall first show that the values $T_\alpha(A)$, $A \in \mathcal{K}^n$, depend only on the Steiner centroid of A for almost all α. Set

$$T'_\alpha(A) = T_\alpha(A) - T_\alpha(\{s(A)\}) + s(A)$$

for $A \in \mathcal{K}^n$, $\alpha \in [0,1]$, and $S'(v) = \int_{(0,1]} T'_\alpha([v]^\alpha)\, d\alpha$ for $v \in \mathcal{F}^n$. Then $T'_\alpha(\{c\}) = c$ for $c \in \mathbb{R}^n$ and S' is a Steiner centroid.

For all $\lambda \in [0,1]$ and $A \in \mathcal{K}^n$, we define a fuzzy set $w_{\lambda,A} \in \mathcal{F}^n$ by $w_{\lambda,A}(s(A)) = 1$, $w_{\lambda,A}(x) = \lambda$ for all $x \in A \setminus \{s(A)\}$, and $w_{\lambda,A}(x) = 0$ for all $x \in \mathbb{R}^n \setminus A$. Thus $[w_{\lambda,A}]^\alpha = A$ for $\alpha \in (0,\lambda]$ and $[w_{\lambda,A}]^\alpha = \{s(A)\}$ for $\alpha \in (\lambda, 1]$. For any fixed $\lambda \in [0,1]$, the function $\mathcal{K}^n \to \mathbb{R}^n$, $A \mapsto S'(w_{\lambda,A})$ fulfils the assumptions of Theorem 2.2, whence $S'(w_{\lambda,A}) = s(A)$. From the definition of S', we obtain

$$\begin{aligned} s(A) &= S'(w_{\lambda,A}) \\ &= \int_{(0,1]}^{\lambda} (T_\alpha(A) - T_\alpha(\{s(A)\}) + s(A))\, d\alpha \\ &\quad + \int_{\lambda}^{1} s(A)\, d\alpha \\ &= \int_0^{\lambda} (T_\alpha(A) - T_\alpha(\{s(A)\}))\, d\alpha + s(A), \end{aligned}$$

hence

$$\int_0^{\lambda} (T_\alpha(A) - T_\alpha(\{s(A)\}))\, d\alpha = 0$$

for all $\lambda \in (0,1]$. Thus $T_\alpha(A) = T_\alpha(\{s(A)\})$ almost everywhere (w.r.t. $\alpha \in (0,1]$).

For each $v \in \mathcal{F}^n$, we define a function $\sigma_v \colon (0,1] \to \mathbb{R}^n$ by $\sigma_v(\alpha) = s([v]^\alpha)$. We proved that

$$\begin{aligned} S(v) &= \int_{(0,1]} T_\alpha(\{s([v]^\alpha)\})\, d\alpha \\ &= \int_{(0,1]} T_\alpha(\{\sigma_v(\alpha)\})\, d\alpha, \end{aligned}$$

thus $S(v)$ is fully determined by σ_v. The mapping $\sigma_. : v \mapsto \sigma_v$ can be associated to a linear mapping $L^2((0,1] \times S^{n-1})^{\mathcal{F}} \to \left(L^2((0,1])\right)^n$. We may express S (considered as a linear mapping $L^2((0,1] \times S^{n-1})^{\mathcal{F}} \to \mathbb{R}^n$) as the composition of σ and a mapping $\phi\colon \left(L^2((0,1])\right)^n \to \mathbb{R}^n$ which is linear, too. Its i-th component, ϕ_i, is a linear functional which may be represented as $\phi_i(f) = \int_{(0,1]} f(\alpha)\,\mu_i(\alpha)\,d\alpha$ for some non-negative function $\mu_i \in L^2((0,1])$ such that $\int_{(0,1]} \mu_i(\alpha)\,d\alpha = 1$. Due to the invariance with respect to Euclidean isometries (in particular, to permutations of coordinates), all μ_i, $i = 1,\dots,n$, must be equal; they coincide to the function μ from the theorem. □

5 Conclusion

In analogy to the crisp case we have called a function from the space $\mathcal{F}^n$ of standard fuzzy sets to $\mathbb{R}^n$ a Steiner centroid if it is invariant under Euclidean isometries, preserves addition, and is continuous w.r.t. the L^2-metric. Unlike in the crisp case, there is more than one Steiner centroid, and we gave a complete description of all Steiner centroids.

Two question remain for further research. The first one is a serious mathematical one. Here, we considered the L^2-metric on $\mathcal{F}^n$. It is not unrealistic, but, as already became clear, tedious to describe Steiner centroids w.r.t. the other metrics $\mathcal{F}^n$ may be equipped with.

The second question concerns possible applications. If the Steiner centroid should really be used, a decision must be made which one to choose. It is at the moment not clear how a particular choice can be motivated.

REFERENCES

[1] D. Butnariu, A.N. Iusem, "Totally Convex Functions for Computation and Infinite Dimensional Optimization", Kluwer Academic Publishers, Dordrecht 2000.

[2] P. Diamond, P. Kloeden, "Metric Spaces of Fuzzy Sets: Theory and Applications", World Scientific, Singapore 1994.

[3] R. Schneider, On Steiner points of convex bodies, *Isr. J. Math.* **9** (1971), 241 - 249.

[4] R. Schneider, "Convex Bodies: the Brunn-Minkowski Theory", Cambridge University Press, Cambridge 1993.

[5] G.C. Shephard, A uniqueness theorem for the Steiner point of a convex region, *J. Lond. Math. Soc.* **43** (1968), 439 - 444.