Partial algebras for Łukasiewicz logics and its extensions

Thomas Vetterlein
Department of computer sciences 1
Dortmund university
44221 Dortmund
Thomas.Vetterlein@uni-dortmund.de

September 2004

Abstract

It is a well-known fact that MV-algebras, the algebraic counterpart of Łukasiewicz logic, correspond to a certain type of partial algebras: lattice-ordered effect algebras fulfilling the Riesz decomposition property. The latter are based on a partial, but cancellative addition, and we may construct from them the representing ℓ-groups in a straightforward manner.

In this paper, we consider several logics differing from Łukasiewicz logics in that they contain further connectives: the PŁ-, PŁ$'$-, PŁ$'_\triangle$-, and ŁΠ-logics. For all their algebraic counterparts, we characterise the corresponding type of partial algebras. We moreover consider the representing f-rings. All in all, we get three-fold correspondences: the total algebras - the partial algebras - the representing rings.

1 Introduction

Although it is actually not easy to say why, it is a clear fact that the Łukasiewicz logic belongs to those calculi which we encounter most frequently among many-valued logics. For a review of recent research, see e.g. [MaMu]. In the last years, several variants of this logic were defined; in particular, the language of Łukasiewicz logic was enriched by further connectives. For instance, the logics PŁ and PŁ$'$ from [HoCi] have an additional connective expressing multiplication; the logic PŁ$'_\triangle$ has, besides this multiplication, a 0-1 projector [Baa, HoCi]; and ŁΠ is the logic combining both Łukasiewicz and product logic [EsGo, EsGoMo].

In this article, we deal with the algebraic counterparts of exactly these logics. The logics PŁ, PŁ$'$, and PŁ$'_\triangle$ were actually designed in a way that they correspond to certain algebras; PMV-, PMV$^+$-, and PMV$_\triangle$-algebras, which were introduced by Montagna in [Mon1, Mon2, Mon3], were taken into account. These algebras, however, were renamed in [HoCi] in accordance with the corresponding logics, and the new notations will also be adopted by us. What we deal with are then the PŁ-, PŁ$'$-, and PŁ$'_\triangle$- algebras. Finally, also ŁΠ-algebras are included in the discussion.

We discuss here the interplay between these (total) algebras on the one hand and the corresponding partial algebras on the other hand. Recall the situation for MV-algebras. MV-algebras may be understood as being based on one total addition-like operation $\oplus$ and a unary complementation-like operation $\sim$. Now, we may, without loss of information, restrict the total addition $\oplus$ to a partial one $+$, which in contrast to the original one is cancellative, and by which also the complementation is easily expressible. The resulting structure – based on a partial addition – is called an effect algebra. Effect algebras have been introduced in a different context, and are actually much more general. The conditions characterising those effect algebras which arise from MV-algebras are the following: they are lattice-ordered and they fulfil a condition which is the analogue of the Riesz decomposition property known for po-groups.

The partial structures are in general more difficult to deal with than with the total ones. But first, we think that it is of interest to have singled out what could be called the "cancellative part" of a total algebra, and to see that the total algebra is reconstructible from the partial one. And second, note that the representing po-groups, or po-rings, are easily constructable from the partial structures. Namely, every MV-algebra L is the interval of an ℓ-group [Mun]; and this representing group is just the group freely generated by the elements of L, subject to the condition $a + b = c$ whenever this equation holds for $a, b, c \in L$ in the corresponding effect algebra.

We proceed as follows. All algebras under consideration are based on a total addition and a product; the partial algebra counterpart is in all cases an effect algebra endowed with a product as an additional operation. This case was studied by Dvurečenskij in [Dvu], where product effect algebras were introduced. We determine the exact properties of the particular product effect algebras which we have under consideration, properties which are not always purely algebraic, but in all cases related to known ones.

PŁ-, PŁ$'$-, PŁ$'_\triangle$-, and ŁΠ-algebras are furthermore representable by partially ordered rings, which are constructible just like in the case of MV-algebras.

Namely, we get f-rings from PŁ-algebras and torsion-free f-rings from PŁ′-algebras [Mon1]. Moreover, we characterise the rings arising from $PŁ'_\triangle$-algebras by a property called strict comparability, an analogue of the general comparability from Goodearl [Goo]. The difficulty to characterise the rings arising from ŁΠ-algebras was pointed out by Montagna [Mon1]; we propose here a property requiring divisibility w.r.t. the multiplication for a certain class of elements.

A schematic summary of the article can be found at the end.

2 MV-algebras

Subject of this paper are the partial algebras and f-rings corresponding to PŁ-, PŁ′-, $PŁ'_\triangle$-, and ŁΠ-algebras. The logics to which they belong all contain Łukasiewicz logic; this is why all the four types of algebras may be understood as MV-algebras enriched by certain further operations.

So in a first step, we recall some basic facts about MV-algebras. A general reference for Łukasiewicz logic and MV-algebras is [CiOtMu]. Axioms which are more standard than those used here can also be found there.

Definition 2.1 An MV-algebra is a structure $(L; \leq, \oplus, \sim, 0, 1)$ with the following properties:

(MV1) $(L; \leq, 0, 1)$ is a lattice with the smallest element 0 and the largest element 1.

(MV2) $(L; \oplus, 0)$ is a commutative semigroup with neutral element 0.

(MV3) $a \leq b$ implies $a \oplus c \leq b \oplus c$ for all a, b, c.

(MV4) $\sim$ is an involutive, order-reversing unary operation.

(MV5) $a \vee b = (a \ominus b) \oplus b$ for all a, b, where

$$a \ominus b \stackrel{\text{def}}{=} \sim(\sim a \oplus b). \qquad (1)$$

Note that MV-algebras are naturally ordered: we have $a \leq b$ iff $a \oplus x = b$ for some $x \in L$.

We will use the following the additional operations on MV-algebras:

$$\begin{aligned} a \otimes b &\overset{\text{def}}{=} \sim(\sim a \oplus \sim b), \\ a \Rightarrow b &\overset{\text{def}}{=} \sim a \oplus b. \end{aligned} \tag{2}$$

$\otimes$ and $\Rightarrow$ then make an MV-algebra L a residuated lattice; for $a, b, c \in L$, we have

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \Rightarrow c. \tag{3}$$

The following is the key definition used in this paper; it tells how to associate with some MV-algebra's total addition a partial one. It is meant to be applicable also to any algebra whose reduct is an MV-algebra.

Definition 2.2 Let $(L; \leq, \oplus, \sim, 0, 1)$ be an MV-algebra. We define the partial operation $+$ on L as follows: For $a, b \in L$, let $a + b = a \oplus b$ if a is the smallest element x such that $x \oplus b = a \oplus b$ and b is the smallest element y such that $a \oplus y = a \oplus b$; else, we let $a + b$ undefined. We call $+$ the *natural partial addition* on L.

The usual way to express the partial $+$ by the total $\oplus$ is as follows; see e.g. [DvPu].

Proposition 2.3 *Let $(L; \leq, \oplus, \sim, 0, 1)$ be an MV-algebra and $+$ the natural partial addition on L. Then $a + b$ is defined if and only if $a \leq \sim b$.*

Effect algebras arose in a completely different context than MV-algebras; they were once introduced to model the internal structure of the unit interval of the po-group of self-adjoint operators of a Hilbert space [FoBe].

Note that, in contrast to the usual definition, we treat here the partial order of an effect algebra as an own relation. For more information about effect algebras see [DvPu].

Definition 2.4 An *effect algebra* is a structure $(L; \leq, +, 0, 1)$ with the following properties:

(E1) $(L; \leq, 0, 1)$ is a poset with a smallest element 0 and a largest element 1.

(E2) $+$ is a partial binary operation such that for any a, b, c

(a) $(a+b)+c$ is defined iff $a+(b+c)$ is defined, and in this case $(a+b)+c = a+(b+c)$;

(b) $a+b$ is defined iff $b+a$ is defined, and in this case $a+b = b+a$.

(c) $a+0$ is always defined and equals a;

(d) $a+b = a+c$ implies $b = c$.

(E3) $a \leq b$ if and only if $b = a+x$ for some x.

We say that an effect algebra $(L; \leq, +, 0, 1)$

(i) is *lattice-ordered* if $(L; \leq)$ is a lattice;

(ii) fulfils the *Riesz decomposition property*, or (RDP) for short, if for all a, b, c, d such that $a+b = c+d$ there are e_1, e_2, e_3, e_4 such that the scheme

$$\begin{array}{cccc} e_1 & e_2 & \rightarrow & a \\ e_3 & e_4 & \rightarrow & b \\ \downarrow & \downarrow & & \\ c & d & & \end{array} \tag{4}$$

holds. Here, by the scheme (4) to hold, we mean that any of the square's column or line adds to what the arrow points to.

An effect algebra which is lattice-ordered and fulfils (RDP), is called an *MV-effect algebra*.

In view of (E3) and (E2)(d), for any a, b such that $a \leq b$ there is a unique x such that $a+x = b$; we will denote this element by $b-a$. Furthermore, we set $\sim a = 1-a$.

In the sequel, any equation involving partial operations reads as usual: It means that all terms are defined and the equation holds.

MV-algebras and MV-effect algebras are related as follows. This connection is, in a slightly modified form, due to [ChKo].

Theorem 2.5 *Let $(L; \leq, \oplus, \sim, 0, 1)$ be an MV-algebra and $+$ the natural partial addition on L. Then $(L; \leq, +, 0, 1)$ is an MV-effect algebra. We then have for $a, b \in L$*

$$a \oplus b = a + (b \wedge \sim a), \tag{5}$$

$$\sim a = \text{the unique } x \text{ such that } a + x = 1. \tag{6}$$

Every MV-effect algebra arises in this way from a unique MV-algebra.

We note that the transition from a total addition to a partial one without loss of information is possible not only for MV-algebras, but also for other algebras arising in fuzzy logics, in particular for (the duals of) BL-algebras [Vet].

The typical examples of effect algebras arise from partially ordered groups.

Definition 2.6 A *unital ℓ-group* is a structure $(G; \leq, +, 0, u)$ such that $(G; \leq, +, 0)$ is an ℓ-group and $u \in G^+$ is a strong unit for G.

Let $(G; \leq, +, 0, u)$ be a unital abelian ℓ-group. We then call $G[0, u] \overset{\text{def}}{=} \{g \in G\colon\ 0 \leq g \leq u\}$ the *unit interval* of G. Define $+$ on $G[0, u]$ as the restriction of the group addition to those pairs of elements whose sum is below u. Then $(G[0, u]; \leq, +, 0, u)$ is a called the *effect algebra arising from G.*

Proposition 2.7 *Let $(G[0, u]; \leq, +, 0, u)$ be the effect algebra arising from some unital abelian ℓ-group G. Then $G[0, u]$ is an MV-effect algebra.*

There is at least one good reason to switch from an MV-algebra to an MV-effect algebra: The ℓ-group representing the MV-algebra may be described in a very simple way. Whereas the fact that MV-algebras all arise from intervals of abelian ℓ-groups is due to Mundici [Mun], the construction of a po-group from an effect algebra was proposed by Ravindran [Rav]. We will describe here the procedure shortly because we have to refer to it in later proofs.

Theorem 2.8 *Let $(L; \leq, +, 0, 1)$ be an MV-effect algebra. Then L is the effect algebra arising from the unit interval of some unital abelian ℓ-group $(\mathcal{G}(L); \leq, +, 0, u)$.*

The unital ℓ-group $\mathcal{G}(L)$ is by L uniquely determined. Moreover, every unital ℓ-group is of the form $\mathcal{G}(L)$ for a unique MV-effect algebra L.

Proof (outlined). Let $(\mathcal{W}(L); +)$ be the commutative semigroup freely generated by the elements of L, subject to the conditions $a = b + c$ whenever this equation holds between elements a, b, c from L. The canonical embedding of L into $\mathcal{W}(L)$ is then injective, whence we may consider L as a subset of $\mathcal{W}(L)$. $(\mathcal{W}(L); +, 0)$ is a commutative, cancellative semigroup with the neutral element 0 and with the property that $a + b = 0$ implies $a = b = 0$.

Moreover, by setting $a \leq b$ in case $a + c = b$ for some c, $\mathcal{W}(L)$ becomes a lattice-ordered semigroup, which has the smallest element 0. We may now identify the interval $\mathcal{W}(L)[0, 1]$, together with the operation $+$ wherever performable and the constants 0 and 1, with the effect algebra $(L; \leq, +, 0, 1)$.

By [Fuc, II, Theorem 4], $\mathcal{W}(L)$ is the positive cone of some po-group $\mathcal{G}(L)$, which then is also lattice-ordered. □

In the sequel, we will, as in this proof, identify any MV-effect algebra L represented by a unital ℓ-group $(\mathcal{G}(L); \leq, +, 0, u)$ *always* with the unit interval of $\mathcal{G}(L)$, that is, we will assume $L = \mathcal{G}(L)[0, u]$.

We summarize that there is a one-to-one correspondence between MV-algebras, MV-effect algebras and unital abelian ℓ-groups. We note that MV-algebras and unital abelian ℓ-groups even form equivalent categories [Mun].

3 PŁ- and PŁ′-algebras

The logics PŁ and PŁ′ were introduced by Horčík and Cintula [HoCi]. They extend Łukasiewicz logic, and they contain a further, multiplication-like connective $\odot$. The algebras belonging to PŁ and PŁ′ are Montagna's PMV- and PMV$^+$-algebras, respectively [Mon1, Mon3]. We will, however, use the terminology of [HoCi], so as to have for the algebras and the corresponding logics the same names.

Definition 3.1 A PŁ-algebra is a structure $(L; \leq, \oplus, \odot, \sim, 0, 1)$ with the following properties:

(PL1) $(L; \leq, \oplus, \sim, 0, 1)$ is an MV-algebra.

(PL2) $(L; \odot, 1)$ is a commutative semigroup with neutral element 1.

(PL3) For a, b, c such that $b \leq \sim c$, we have

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c).$$

Moreover, a PŁ′-algebra is a PŁ-algebra such that:

(PL4) For any a, $a \odot a = 0$ implies $a = 0$.

In [HoCi], distributivity of $\odot$ over $\ominus$ is postulated rather than over $\oplus$; note from the following lemma that this makes no difference. Recall that $\ominus$ is defined by (1).

Lemma 3.2 A structure $(L; \leq, \oplus, \odot, \sim, 0, 1)$ is a PŁ-algebra if and only if (PL1), (PL2), and (PL4) hold as well as:

(PL3') For any a, b, c, we have

$$a \odot (b \ominus c) = (a \odot b) \ominus (a \odot c). \tag{7}$$

Proof. Assume that L is a PŁ-algebra. Note first that $a \odot (b \vee c) = a \odot [((b \wedge c) \oplus (b \ominus c)) \vee ((b \wedge c) \oplus (c \ominus b))] = a \odot ((b \wedge c) \oplus (b \ominus c) \oplus (c \ominus b)) = (a \odot (b \wedge c)) \oplus (a \odot (b \ominus c)) \oplus (a \odot (c \ominus b)) = [(a \odot (b \wedge c)) \oplus (a \odot (b \ominus c))] \vee [(a \odot (b \wedge c)) \oplus (a \odot (c \ominus b))] = (a \odot b) \vee (a \odot c)$.

So we may conclude $(a \odot (b \ominus c)) \oplus (a \odot c) = a \odot (b \vee c) = (a \odot b) \vee (a \odot c) = ((a \odot b) \ominus (a \odot c)) \oplus (a \odot c)$, which implies (PL3').

Conversely, assume (PL1), (PL2), (PL3'), and (PL4) to hold. Note that $\odot$ is then isotone. If now $b \leq \sim c$, we have $(b \oplus c) \ominus c = b$, hence $(a \odot (b \oplus c)) \ominus (a \odot c) = a \odot b$, and (PL3) follows. □

When we now restrict the total operation of a PŁ-algebra to a partial one, just like in the case of MV-algebras, we arrive at effect algebras with one further operation, $\odot$. In [Dvu], product effect algebras were introduced, which fit exactly into the present context. Product effect algebras, however, are more general structures than those we will define now; see the remark below.

Definition 3.3 An *f-product effect algebra* is a structure $(L; \leq, +, \odot, 0, 1)$ with the following properties:

(PE1) $(L; \leq, +, 0, 1)$ is an MV-effect algebra.

(PE2) $(L; \odot, 1)$ is a commutative semigroup with neutral element 1.

(PE3) For a, b, c such that $b + c$ is defined, also $(a \odot b) + (a \odot c)$ is defined, and

$$a \odot (b + c) = (a \odot b) + (a \odot c).$$

Moreover, an f-product effect algebra is called *torsion-free* if, for all a, $a \odot a = 0$ implies $a = 0$.

Remark 3.4 A product effect algebra according to [Dvu] is a structure $(L; \leq, +, \odot, 0, 1)$ such that $(L; \leq, +, 0, 1)$ is an effect algebra and $\odot$ distributes over $+$ from both sides. So lattice order, (RDP), and the axiom (PL2) is what is assumed more here.

In later sections, we need the following facts.

Lemma 3.5 *Let $(L; \leq, +, \odot, 0, 1)$ be an f-product effect algebra.*

(i) *For any $a, b, c \in L$, $a \leq b$ implies $a \odot c \leq b \odot c$.*
In particular, for any a, c we have $a \odot c \leq a$.

(ii) *For any $a, b, c \in L$, $(a \wedge b) \odot c = (a \odot c) \wedge (b \odot c)$ and $(a \vee b) \odot c = (a \odot c) \vee (b \odot c)$.*

(iii) *L is torsion-free if and only if, for any a and any $k \geq 1$, $a^k = 0$ implies $a = 0$. Here, $a^k \stackrel{\text{def}}{=} \underbrace{a \odot \ldots \odot a}_{k \text{ times}}$.*

(iv) *If L is torsion-free, then $a \odot b = 0$ implies $a \wedge b = 0$.*

Proof. (i) If $a \leq b$, we may multiply $a + (b - a) = b$ by c on both sides.

(ii) Let $a' = a - (a \wedge b)$ and $b' = b - (a \wedge b)$; then $a' \wedge b' = 0$. Note that in lattice-ordered effect algebras, $+$ distributes over $\wedge$ if the involved sums exist; so $(a \odot c) \wedge (b \odot c) = [((a \wedge b) + a') \odot c] \wedge [((a \wedge b) + b') \odot c] = [(a \wedge b) \odot c] + [(a' \odot c) \wedge (b' \odot c)] = (a \wedge b) \odot c$, where we made use of (i) in the last step.

This is the first part, from which the second part follows by $a \vee b = a + [b - (a \wedge b)]$.

(iii) By the second part of (i), $a^k = 0$ for some a and $k \geq 1$, implies $a^{k'}$ for some power $k' \geq k$ of 2.

(iv) If $a \odot b = 0$, then $(a \wedge b) \odot (a \wedge b) = 0$ by (i), so $a \wedge b = 0$ if L is torsion-free. □

The basic correspondence between f-product effect algebras on the one hand and PŁ-algebras on the other hand, is described next.

Theorem 3.6 *Let* $(L; \leq, \oplus, \odot, \sim, 0, 1)$ *be a* PŁ*-algebra and* $+$ *the natural partial addition on* L*. Then* $(L; \leq, +, \odot, 0, 1)$ *is an f-product effect algebra.* $\oplus$*,* $\sim$ *are reobtained by* (5), (6)*.*

Every f-product effect algebra arises in this way from a unique PŁ*-algebra.*

Under this correspondence, the PŁ$'$*-algebras are exactly the torsion-free f-product effect algebras.*

Proof. This follows from Theorem 2.5 and is otherwise an easy check of the respective axioms. □

The typical examples of f-product effect algebras arise from partially ordered rings. Note from the following definition that here all rings are assumed to be commutative, associative, and with 1.

Definition 3.7 A *ring* is a structure $(R; +, \odot, 0, 1)$ such that $(R; +, 0)$ is an abelian group with neutral element 0, $(R; \odot, 1)$ is a commutative semigroup with neutral element 1, and $\odot$ distributes over $+$.

A structure $(R; \leq, +, \odot, 0, 1)$ is called a *po-ring* if $(R; +, \odot, 0, 1)$ is a ring and $\leq$ is a partial order such that (i) $(R; \leq, +, 0)$ is a po-group and (ii) $a \leq b$ implies $a \odot c \leq b \odot c$ for $c \geq 0$. R is called *unital* if in addition $(R; \leq, +, 0, 1)$ is a unital po-group.

Moreover, an *f-ring* is a po-ring R such that (i) R is lattice-ordered and (ii) $a \wedge b = 0$ implies $(a \odot c) \wedge b = 0$ for any $a, b, c \in R$, $c \geq 0$. A ring is called *torsion-free* if it does not have any non-zero nilpotent element.

Let $(R; \leq, +, \odot, 0, 1)$ be a unital f-ring. We again call $R[0,1] \stackrel{\text{def}}{=} \{g \in R\colon\ 0 \leq g \leq 1\}$ the *unit interval* of R. Let $(R[0,1]; \leq, +, \odot, 0, 1)$ be the effect algebra arising from the unital ℓ-group $(R; \leq, +, 0, 1)$, endowed with the further operation $\odot$. Then $R[0,1]$ is a called the *product effect algebra arising from R*.

Proposition 3.8 *Let* $(R[0,1]; \leq, +, \odot, 0, 1)$ *be a product effect algebra arising from some unital f-ring* R*. Then* $R[0,1]$ *is an f-product effect algebra.*

In [Dvu], it is proved that any product effect algebra fulfilling (RDP) arises from the interval of a po-group, in the sense of Definition 3.7. We give here a version of this representation theorem adapted to the more special conditions we have to do with here.

Theorem 3.9 *Let $(L; \leq, +, \odot, 0, 1)$ be an f-product effect algebra. Then L is the product effect algebra arising from some unital f-ring $(\mathcal{R}(L); \leq, +, \odot, 0, 1)$.*

$\mathcal{R}(L)$ is by L uniquely determined. Moreover, every unital f-ring is of the form $\mathcal{R}(L)$ for a unique f-product effect algebra L.

An f-product effect algebra is torsion-free if and only if so is the corresponding f-ring.

Proof (first part summarized from [DiDv, Dvu]; for details, see there). As in the proof of Theorem 2.8, let $(\mathcal{W}(L); \leq, +, 0)$ be the semigroup freely generated by L, subject to $a + b = c$ if this holds in L, and let $\mathcal{W}(L)$ be endowed with the natural order. Note that $\mathcal{W}(L)$ has the Riesz decomposition property (RDP), understood in the obvious way.

Now, for any $a, b \in \mathcal{W}(L)$, set $a \odot b = \sum_{\substack{i=1,\ldots,m \\ j=1,\ldots n}} a_i \odot b_j$, where $a = a_1 + \ldots + a_m$ and $b = b_1 + \ldots + b_n$ are written as sums of elements of L; using (RDP), we see that $\odot$ is defined in this way unambigously. Then $(\mathcal{W}(L); \odot, 1)$ is a commutative semigroup with the neutral element 1; $\odot$ distributes over $+$; and $a \leq b$ implies $a \odot c \leq b \odot c$ for all $a, b, c \in \mathcal{W}(L)$. In particular, $(\mathcal{W}(L); +, \odot, 0, 1)$ is, in the sense of [Fuc], a conic semiring.

By [Fuc, VI, Theorem 2], $\mathcal{W}(L)$ is the positive cone of some po-ring $(\mathcal{R}(L); +, \odot, 0, 1, \leq)$. $\mathcal{R}(L)$ is then lattice-ordered, and, using (RDP), it is not difficult to see that $\mathcal{R}(L)$ is even an f-ring. So the first statement of the theorem is proved.

Concerning the second part, note that by Theorem 2.8, the unital ℓ-group underlying $\mathcal{R}(L)$ is uniquely determined by the effect algebra $(L; \leq, +, 0, 1)$, and the operation $\odot$ is by distributivity uniquely determined by its restriction to L. Moreover, from Proposition 3.8 and Theorem 2.8 we conclude that all unital f-rings may be constructed, in the way shown, from its unit intervals.

Assume now that L is torsion-free. For some $a \in \mathcal{W}(L)$, let $a^k = 0$ for some $k \geq 1$. Write $a = a_1 + \ldots + a_n$ as a sum of elements of L; by distributivity and the fact that if in $\mathcal{W}(L)$ a sum is 0, all summands are 0, it follows $a_i^k = 0$, hence by Lemma 3.5(iii) $a_i = 0$ for all $i = 1, \ldots, n$; so $a = 0$.

Let now $a \in \mathcal{R}(L)$ such that $a^k = 0$. We then have $|a|^k = |a^k| = 0$ by [Bir, XVII,§5], thus $|a| = 0$ and $a = 0$. The proof is complete that nilpotent elements in $\mathcal{R}(L)$ must be 0, that is, $\mathcal{R}(L)$ is torsion-free.

It is clear that if $\mathcal{R}(L)$ is torsion-free, then so is L. □

So there is one-to-one-correspondence (i) between PŁ-algebras, f-product effect algebras, and unital f-rings, and (ii) between PŁ'-algebras, torsion-free f-product effect algebras, and unital torsion-free f-rings.

4 $\text{PŁ}'_\triangle$-algebras

In [Baa], Baaz proposed a fuzzy logic whose language contains a unary connective denoted by the symbol $\triangle$; when evaluating a formula's truth value, it maps all values from the real unit interval strictly smaller than 1 to 0, and 1 itself to 1. $\text{PŁ}'_\triangle$ is the logic which extends $\text{PŁ}'$ in that it contains the additional connective $\triangle$, interpreted in the indicated way. For details, we refer to [HoCi].

We note that it is surely also possible to extend the Łukasiewicz- of the PŁ-logic by the Baaz $\triangle$-connective, to deal then with $\text{MV}_\triangle$- or $\text{PŁ}_\triangle$-algebras, respectively. See the concluding remarks at the end of this paper.

The algebras corresponding to the logic $\text{PŁ}'_\triangle$ are the following. They coincide with Montagna's $\text{PMV}_\triangle$-algebras [Mon2].

Definition 4.1 A $\text{PŁ}'_\triangle$-algebra is a structure $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ such that $(L; \leq, \oplus, \odot, \sim, 0, 1)$ is a $\text{PŁ}'$-algebra and such that the following holds:

(PLb) $\triangle$ is a unary operation such that for any a, b

(a) $\triangle a \leq a$,

(b) $\triangle\triangle a = \triangle a$,

(c) $\triangle 1 = 1$,

(d) $\triangle a \vee \sim\triangle a = 1$,

(e) $\triangle(a \vee b) = \triangle a \vee \triangle b$,

(f) $\triangle(a \oplus b) \;\leq\; \nabla a \oplus \triangle b$,

where $\nabla \colon L \to L, \quad a \mapsto \sim\triangle\sim a$.

We will further use the operations $\Rightarrow$ and $\otimes$ defined according to (2). Note that condition (PLb)(f) may also be written as

$$\triangle(a \Rightarrow b) \;\leq\; \triangle a \Rightarrow \triangle b \tag{8}$$

for any a, b.

Lemma 4.2 *Let* $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ *be a* $\text{PŁ}'_\triangle$*-algebra. Then for any* $a, b \in L$ *we have*

(i) $a = \triangle a$ *if and only if* $a \wedge \sim a = 0$.

(ii) $a \otimes \triangle b = a \wedge \triangle b$.

(iii) $\triangle(a \Rightarrow b) \vee \triangle(b \Rightarrow a) = 1$ *and* $\sim\triangle(a \Rightarrow b) \leq \triangle(b \Rightarrow a)$.

Proof. (i) If $a = \triangle a$, then $a \vee \sim a = 1$ by (PLb)(d), hence also $a \wedge \sim a = 0$. If $a \wedge \sim a = 0$, then $1 = a \vee \sim a = \triangle(a \vee \sim a) = \triangle a \vee \triangle \sim a$ by (PLb)(c),(e), so $a = (a \wedge \triangle a) \vee (a \wedge \triangle \sim a) = \triangle a$ by (PLb)(a).

(ii) By (PLb)(d), we have $a = a \otimes (\triangle b \vee \sim\triangle b) = (a \otimes \triangle b) \vee (a \otimes \sim\triangle b)$, so $a \wedge \triangle b = a \otimes \triangle b$.

(iii) $(a \Rightarrow b) \vee (b \Rightarrow a) = 1$ holds in MV-algebras. So (PLb)(c),(e) imply the first part, from which the second easily follows by (PLb)(d). □

We now turn back to partial algebras. The partial algebra analogue for $\text{PŁ}'_\triangle$-algebras will not be f-product effect algebras endowed with an operation $\triangle$. We shall rather use f-product effect algebras themselves, requiring that there are sufficiently many sharp elements. The latter, which sometimes are also called *boolean*, are to be defined next.

Definition 4.3 Let $(L; \leq, +, \odot, 0, 1)$ be an f-product effect algebra. We call $e \in L$ *sharp* if $e \wedge \sim e = 0$ and $e \vee \sim e = 1$; we denote the set of sharp elements of L by $S(L)$.

Lemma 4.4 *Let* $(L; \leq, +, \odot, 0, 1)$ *be an f-product effect algebra.*

(i) $(S(L); \leq, 0, 1)$ *is a boolean lattice. The complementation is given by* $\sim$*; and for* $e, f \in S(L)$ *such that* $e \wedge f = 0$*, we have* $e \vee f = e + f$.

(ii) *For any* $a \in L$ *and* $e, f \in S(L)$ *such that* $e \wedge f = 0$ *and* $a \leq e + f$*, we have* $a = a \wedge e + a \wedge f$.

(iii) *For* $a \in L$ *and* $e \in S(L)$*, we have* $a \odot e = a \wedge e$.

Proof. (i), (ii) are easily checked; cf. also [GrFoPu].

(iii) We have $a \odot e \leq a \wedge e$ and $(a \wedge e) - (a \odot e) = (a - a \odot e) \wedge (e - a \odot e) = (a \odot \sim e) \wedge (e \odot \sim a) \leq \sim e \wedge e = 0$. □

Now, the condition which we propose to characterise those f-product effect algebras which correspond to $\text{PŁ}'_\triangle$-algebras, is a strengthened version of an analogue of a condition known for po-groups, namely general comparability from [Goo]; cf. Definition 4.11(i) below.

Definition 4.5 Let $(L; \leq, +, \odot, 0, 1)$ be an effect algebra. We say that L fulfils *strict comparability* if for any pair $a, b \in L$, there is a sharp element e such that $a \wedge e \leq b \wedge e$ and, for any non-zero sharp element $f \leq {\sim}e$, $a \wedge f > b \wedge f$. In the sequel, we will denote an element e with these properties by e_a^b.

In fact, e_a^b will denote *the* element associated to a pair of elements a and b as specified in Definition 4.5:

Lemma 4.6 *Let $(L; \leq, +, \odot, 0, 1)$ be an effect algebra fulfilling strict comparability. Then for any pair a, b, the element e_a^b specified in Definition 4.5 is uniquely determined.*

Proof. Given $a, b \in L$ and two distinct elements $e_1, e_2 \in S(L)$ satisfying the inequalities from Definition 4.5, then there is, possibly after interchanging e_1 with e_2, some non-zero sharp $f \leq e_2, {\sim}e_1$, which would mean $a \wedge f \leq b \wedge f$ and $a \wedge f > b \wedge f$. □

Lemma 4.7 *Let $(L; \leq, +, \odot, 0, 1)$ be an f-product effect algebra fulfilling strict comparability.*

(i) *For any $a, b \in L$, we have $e_a^b \vee e_b^a = 1$.*

(ii) *For every $a \in L$, there is a smallest sharp element $\overline{a} \geq a$. We have $\overline{a} = {\sim}e_a^0$.*

(iii) *If $a \wedge b = 0$ for a pair $a, b \in L$, then $\overline{a} \wedge \overline{b} = 0$ also for the smallest sharp elements $\overline{a} \geq a$ and $\overline{b} \geq b$.*

Proof. (i) If $e_a^b \vee e_b^a < 1$, then there is a non-zero sharp $f \leq {\sim}e_a^b, {\sim}e_b^a$; a contradiction.

(ii) Let $\overline{a} = {\sim}e_a^0$; then $\overline{a} \in S(L)$, $a \wedge {\sim}\overline{a} = 0$ and $a \wedge f > 0$ for all non-zero sharp $f \leq \overline{a}$. So $\overline{a} \geq a$, and if $a \leq f \leq \overline{a}$ for some sharp element f, then

$(\overline{a} - f) \wedge a = 0$ because $[(\overline{a} - f) \wedge a] + (f \wedge a) = a$ by Lemma 4.4(ii); thus $f = \overline{a}$.

(iii) Let $e = e_a^b$. We then have $a \wedge e \leq b \wedge e$ and $a \wedge {\sim}e > b \wedge {\sim}e$. From $a \wedge b = 0$, it follows $a \wedge e = b \wedge {\sim}e = 0$ and further $a \leq {\sim}e$, $b \leq e$. So $a \leq \overline{a} \leq {\sim}e$ and $b \leq \overline{b} \leq e$, and $\overline{a} \wedge \overline{b} = 0$. □

Theorem 4.8 *Let* $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ *be a* $\text{PŁ}'_\triangle$*-algebra and* $+$ *the natural partial addition on* L*. Then the f-product effect algebra* $(L; \leq, +, \odot, 0, 1)$ *fulfils strict comparability.* $\oplus$*,* $\sim$ *are reobtained by* (5), (6)*; and* $\triangle = {\sim}\nabla{\sim}$*, where* $\nabla a = \overline{a}$*, that is, the smallest sharp element above some* $a \in L$*.*

Every torsion-free f-product effect algebra fulfilling strict comparability arises in this way from a unique $\text{PŁ}'_\triangle$*-algebra.*

Proof. Let $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ be a $\text{PŁ}'_\triangle$-algebra; $+$ being its natural partial addition, we have by Theorem 3.6 that $(L; \leq, +, \odot, 0, 1)$ is a torsion-free f-product effect algebra and that (5), (6) hold.

To see that L has strict comparability, let $a, b \in L$, and set $e = \triangle(a \Rightarrow b)$. From $e \leq a \Rightarrow b$ we have $a \otimes e \leq b$ by (3), so $a \wedge e \leq b \wedge e$ by Lemma 4.2(ii). On the other hand, ${\sim}\triangle(a \Rightarrow b) \leq \triangle(b \Rightarrow a)$ by Lemma 4.2(iii), hence ${\sim}\triangle(a \Rightarrow b) \leq b \Rightarrow a$ and $b \wedge {\sim}\triangle(a \Rightarrow b) \leq a$; so $a \wedge {\sim}e \geq b \wedge {\sim}e$. Suppose $a \wedge f = b \wedge f$ for some sharp element $f \leq {\sim}e$; then $a \otimes (e \vee f) = a \wedge (e \vee f) = (a \wedge e) \vee (a \wedge f) \leq (b \wedge e) \vee (b \wedge f) \leq b$, so $f \leq e \vee f \leq a \Rightarrow b$, whence $f = \triangle f \leq \triangle(a \Rightarrow b) = e$, that is, $f = 0$. So strict comparability is proved, and $e_a^b = e = \triangle(a \Rightarrow b)$.

It moreover follows $\nabla a = {\sim}\triangle{\sim}a = {\sim}\triangle(a \Rightarrow 0) = {\sim}e_a^0$, which by Lemma 4.7(ii) is the smallest sharp element above a.

For the converse direction, let $(L; \leq, +, \odot, 0, 1)$ be a torsion-free f-product effect algebra fulfilling strict comparability. Let $(L; \leq, \oplus, \odot, \sim, 0, 1)$ be the $\text{PŁ}'$-algebra belonging to it according to Theorem 3.6. Furthermore, for any a, we define $\triangle a = {\sim}\,\overline{{\sim}a}$. We have to verify (PLb)(a)–(f).

Clearly, $\overline{a} \geq a$, $\overline{\overline{a}} = \overline{a}$, $\overline{0} = 0$, so (PL2)(a)–(c) follow; and (PL2)(d) holds because $\overline{a}$ is a sharp element.

We next show $\overline{a} \wedge \overline{b} = \overline{a \wedge b}$, from which (PL2)(e) will follow. Indeed, $\overline{a \wedge b} \leq \overline{a} \wedge \overline{b}$, and if there is a sharp $f \leq \overline{a}, \overline{b}$ such that $f \wedge (\overline{a \wedge b}) = 0$, we conclude $f \wedge a \wedge b = 0$ and, applying twice Lemma 4.7(iii) gives $f = f \wedge \overline{a} \wedge \overline{b} = 0$.

We finally see that $\overline{b} \ominus \overline{a} \leq \overline{b \ominus a}$, implying (PL2)(f) by (8). Indeed, we have $b \ominus a \leq \overline{b \ominus a}$, so by (3) $b \otimes \sim(\overline{b \ominus a}) \leq a$, and by Lemma 4.2(ii) $b \wedge \sim(\overline{b \ominus a}) \leq a$. Applying $\bar{\cdot}$ on both sides gives $\overline{b} \wedge \sim(\overline{b \ominus a}) \leq \overline{a}$; this in turn implies $\overline{b} \ominus \overline{a} \leq \overline{b \ominus a}$. □

We will now turn to the f-rings whose unit intervals give rise to PŁ$'_\triangle$-algebras. The following definitions show where the notion of strict comparability comes from – see [Goo, Chapter 8]. The notions now introduced will certainly also be used for the ℓ-groups underlying f-rings.

Definition 4.9 Let $(G; \leq, +, 0, u)$ be a unital abelian ℓ-group. We call an element $e \in G$ such that $0 \leq e \leq u$ *characteristic* if $e \wedge (u - e) = 0$ and $e \vee (u - e) = u$; we denote the set of characteristic elements of L by $S(G)$.

From [Goo], we have:

Lemma 4.10 *Let $(G; \leq, +, 0, u)$ be a unital abelian ℓ-group.*

(i) *$(S(G); \leq, 0, u)$ is a boolean lattice.*

(ii) *Let e be a characteristic element of G, and let G_e and G_{u-e} be the convex subgroups of G generated by e and $u - e$, respectively. Then $G_e \cap G_{u-e} = \{0\}$, and $G = G_e + G_{u-e}$. In particular, any $a \in G$ may be uniquely written as $a = p_e(a) + p_{u-e}(a)$, where $p_e(a) \in G_e$ and $p_{u-e}(a) \in G_{u-e}$.*

Assuming $-nu \leq a \leq nu$, we have

$$p_e(a) = (a^+ \wedge ne) - ((-a)^+ \wedge ne). \tag{9}$$

In the sequel, for an characteristic element e of an ℓ-group, p_e will denote the function according to Lemma 4.10.

Definition 4.11 Let $(G; \leq, +, 0, u)$ be a unital abelian ℓ-group.

(i) We say that G fulfils *general comparability* if for any pair $a, b \in L$, there is a characteristic element $e \in L$ such that $p_e(a) \leq p_e(b)$ and $p_{u-e}(a) \geq p_{u-e}(b)$.

(ii) We say that G fulfils *strict comparability* if for any pair $a, b \in L$, there is a characteristic element $e \in L$ such that $p_e(a) \leq p_e(b)$ and, for any non-zero characteristic element f such that $e \wedge f = 0$, $p_f(a) > p_f(b)$.

Clearly, strict comparability implies general comparability. We moreover note:

Lemma 4.12 *Let* $(G; \leq, +, 0, u)$ *be a unital abelian* ℓ*-group. Then* G *fulfils strict comparability if and only for any pair of positive elements* $a, b \in G$*, there is an* e *as specified in Definition* 4.11*.*

Proof. Let $a \in G$. For some $n \geq 1$, we have $-nu \leq a \leq nu$. By Lemma 4.10(ii), $p_e(a + nu) = p_e(a) + p_e(nu) = p_e(a) + ne$ for any $e \in S(G)$, and the assertion follows. □

Theorem 4.13 *Let* $(L; \leq, +, \odot, 0, 1)$ *be an f-product effect algebra. Then* L *fulfils strict comparability if and only if so does the representing f-ring* $(\mathcal{R}(L); \leq, +, \odot, 0, 1)$*.*

Proof. The infimum or supremum of a pair of elements of L is the same when calculated in L or in $\mathcal{R}(L)$. So in particular, characteristic elements of $\mathcal{R}(L)$ are exactly the sharp elements of L.

Now, if $\mathcal{R}(L)$ fulfils strict comparability, it easily follows from (9) that L fulfils strict comparability, too.

Conversely, let L fulfil strict comparability. We first prove that, for any $a \in \mathcal{R}(L)$, there is an $e \in S(L)$ such that $p_e(a) \geq 0$ and $p_f(a) < 0$ for all non-zero $f \in S(L)$ such that $f \leq \sim e$. We have $a = a^+ - (-a)^+$, where $a^+, (-a)^+ \geq 0$ and $a^+ \wedge (-a)^+ = 0$; set $e = \sim(\overline{(-a)^+ \wedge u})$; then $e \geq \overline{a^+ \wedge u}$ by Lemma 4.7(iii). Let n be such that $-nu \leq a \leq nu$. So then $a^+ \leq nu$, whence $0 \leq a^+ \leq n(a^+ \wedge u) \leq ne$; and similarly $0 \leq (-a)^+ \leq n(\sim e)$; so $p_e(a) = a^+ \geq 0$ and $p_{\sim e}(a) = -(-a)^+ \leq 0$. Furthermore, for any sharp $f \leq \sim e$, we have by Lemma 4.10(ii) $p_f(a) = -((-a)^+ \wedge nf) \leq 0$; $p_f(a) = 0$ then means $(-a)^+ \wedge f = 0$, so $f = \sim e \wedge f = (\overline{(-a)^+ \wedge u}) \wedge f = 0$. We note that $a \ngeq 0$ holds exactly in case $\sim e > 0$.

In view of (the proof of) Lemma 4.12, it further follows that, given any $a \in \mathcal{R}(L)$ and $i \geq 1$, there is an $e_i \in S(L)$ such that $p_{e_i}(a) \geq ie_i$ and $p_f(a) < if$ for any non-zero sharp $f \leq \sim e_i$. Let now $n \geq 2$ be such that $0 \leq a \leq (n-1)u$; let $g_0 = \sim e_1$, $g_i = e_i \wedge \sim e_{i+1}$ for $i = 1, \ldots, n-2$, $g_{n-1} = e_{n-1}$. Then $g_i \wedge g_j = 0$ if $i \neq j$, $g_0 + \ldots + g_{n-1} = u$, and $ig_i \leq p_{g_i}(a) < (i+1)g_i$ for all $i = 1, \ldots, n-1$ such that $g_i > 0$.

Let now $a, b \in \mathcal{R}(L)$ and, in view of Lemma 4.12, assume $a, b \geq 0$. Choose $n \geq 2$ such that $a, b \leq (n-1)u$, and apply the result of the last paragraph to

b, so as to get $g_0, \ldots, g_{n-1}$ such that $b = p_{g_0}(b) + \ldots + p_{g_{n-1}}(b)$ and $ig_i \leq p_{g_i}(b)$ as well as $p_f(b) < (i+1)f$ for a non-zero sharp $f \leq g_i$; $i = 0, \ldots, n-1$. Let $b_i = p_{g_i}(b) - ig_i$, so that $b_i \in L$; and let $a_i = p_{g_i}(a) - ig_i$ and $a'_i = a_i^+ \wedge g_i$, so that also $a'_i \in L$.

By strict comparability of L, for every $i = 0, \ldots, n-1$ such that $g_i > 0$, there is an $e_i \leq g_i$ such that $a'_i \wedge e_i \leq b_i \wedge e_i$ and $a'_i \wedge f > b_i \wedge f$ for any non-zero sharp $f \leq {\sim}e_i \wedge g_i$. We shall show $p_{e_i}(a_i) \leq p_{e_i}(b_i)$ and $p_f(a_i) > p_f(b_i)$; Lemma 4.12 will then imply that the same inequalities hold also for a and b replacing a_i and b_i, respectively; and so strict comparability of $\mathcal{R}(L)$ will become easily derivable.

Now, there cannot be any non-zero sharp $f \leq e_i$ such that $p_f(a_i^+) \geq f$, because in this case $a_i^+ \geq f$, contradicting $a_i^+ \wedge f = a'_i \wedge f \leq b_i \wedge f = p_f(b_i) < f$. It follows $p_{e_i}(a_i^+) \leq e_i$, and so $p_{e_i}(a_i) \leq p_{e_i}(a_i^+) = a_i^+ \wedge e_i = a'_i \wedge e_i \leq b_i \wedge e_i = p_{e_i}(b_i)$.

Furthermore, let $f \leq {\sim}e_i \wedge g_i$ be a non-zero sharp element. We have $(a_i \wedge f) \vee 0 > b_i \wedge f \geq 0$; let us assume $a_i \wedge f \not> b_i \wedge f$. It follows $a_i \wedge f \not\geq 0$, and so there is a non-zero sharp $f' \leq f$ such that $p_{f'}(a_i \wedge f) < 0$. But this means $a_i \wedge f' < 0$ and $a'_i \wedge f' = (a_i \wedge f') \vee 0 = 0$, contradicting $a'_i \wedge f' > b_i \wedge f' \geq 0$. □

Taking also into account also Theorem 3.9, we have thus derived that there is a one-to-one correspondence between PŁ$'_\triangle$-algebras, torsion-free f-product effect algebras with strict comparability, and unital torsion-free f-rings with strict comparability.

5 ŁΠ-algebras

The logic ŁΠ was introduced in [EsGo] as a combination of Łukasiewicz and product logic. We understand this logic as the product logic enriched by the Łukasiewicz negation; this is why we will axiomatize the corresponding algebras, the ŁΠ-algebras, in a little bit uncommon way. On the other hand, our axioms are very close to Cintula's in [Cin], and the fact that both systems coincide is seen in Proposition 5.5 below.

Product logic is an extension of Hájek's Basic fuzzy logic; the corresponding algebras are special BL-algebras. For these items, we refer to [Haj]. For more information about ŁΠ-algebras, we refer to [EsGoMo, Mon1, MoPa, Cin].

Definition 5.1 A product algebra is a structure $(L; \leq, \odot, \rightarrow, 0, 1)$ with the following properties:

(LP1) $(L; \leq, \odot, \rightarrow, 0, 1)$ is a residuated lattice.

(LP2) For any a, b we have:

(a) $a \leq b$ implies that $a = b \odot k$ for some k,

(b) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

(LP3) Let $\neg a \stackrel{\text{def}}{=} a \rightarrow 0$. For any a, b, c, we have:

(a) $a \wedge \neg a = 0$,

(b) if $\neg\neg a = 1$, then $a \odot b = a \odot c$ implies $b = c$.

An ŁΠ-algebra is a structure $(L; \leq, \odot, \rightarrow, \sim, 0, 1)$ such that $(L; \leq, \odot, \rightarrow, 0, 1)$ is a product algebra and such that the following holds:

(LP4) $\sim$ is a order-reversing, involutive unary operation, which furthermore satisfies for all a:

(a) $\neg a \leq \sim a$,

(b) $\sim\neg a = \neg\neg a$.

(LP5) For all a, b, $a \otimes b = b \otimes a$, where

$$a \otimes b \stackrel{\text{def}}{=} a \odot \sim(a \rightarrow \sim b). \tag{10}$$

We will identify ŁΠ-algebras, as defined here, with a subclass of the $\text{PŁ}'_\triangle$-algebras. This result will then allow us to characterise the underlying partial structure; the announced equivalence of our axioms with those in [Cin] will be an immediate corrolary.

Definition 5.2 We call a $\text{PŁ}'_\triangle$-algebra $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ *divisible* if for any a, b such that $a \leq b$, there is some k such that $b \odot k = a$.

Lemma 5.3 *Let $(L; \leq, \odot, \rightarrow, 0, 1)$ be a product algebra. Then we have for all a, b, c*

$$a \odot b \rightarrow a \odot c = (b \rightarrow c) \vee \neg a. \tag{11}$$

In particular, $a \rightarrow a \odot b = b \vee \neg a$.

Proof. Assume first $\neg\neg a = 1$. Then, by (LP3)(b), $a \odot c = a \wedge (a \odot c) = a \odot (a \to a \odot c)$, so $a \to a \odot c = c$. It also follows $a \odot b \to a \odot c = b \to (a \to a \odot c) = b \to c$.

Now, let a be any element, and set $a' = a \vee \neg a$. Then $\neg\neg a' = 1$ by (LP3)(a), and $b \to c = a' \odot b \to a' \odot c \geq a' \odot b \to a \odot c = (a \odot b \to a \odot c) \wedge (\neg a \odot b \to a \odot c)$. Because $\neg a \leq a \odot b \to a \odot c$ and $(\neg a \odot b \to a \odot c) \vee \neg a = (\neg a \to (b \to a \odot c)) \vee \neg a \geq \neg\neg a \vee \neg a = 1$, we conclude $(b \to c) \vee \neg a \geq a \odot b \to a \odot c$. Clearly, $(b \to c) \vee \neg a \leq a \odot b \to a \odot c$. □

Theorem 5.4 *Let $(L; \leq, \odot, \to, \sim, 0, 1)$ be an ŁΠ-algebra, and let for $a, b \in L$*

$$a \oplus b = \sim(\sim a \otimes \sim b), \quad (12)$$

$$\triangle a = \neg \sim a, \quad (13)$$

where $\otimes$ is defined by (10)*. Then $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ is a divisible* $\text{PŁ}'_\triangle$*-algebra. We reobtain $\to$ by*

$$a \to b = \max\{x \colon\ a \odot x \leq b\}. \quad (14)$$

Moreover, every divisible $\text{PŁ}'_\triangle$*-algebra arises in this way from unique ŁΠ-algebra.*

Proof (of the first part). Let $(L; \leq, \odot, \to, \sim, 0, 1)$ be an ŁΠ-algebra. Note that then, in particular, $(L; \leq, \odot, \to, 0, 1)$ is a BL-algebra.

First, we prove that $(L; \leq, \oplus, \sim, 0, 1)$ is an MV-algebra. (MV1) holds by (LP1). Moreover, $\oplus$ is commutative by (LP5). 0 is a neutral element w.r.t. $\oplus$; to see this, put $a = 1$ in (10). By the commutativity of $\otimes$, we have $a \otimes (b \otimes c) = (b \otimes c) \otimes a = b \odot \sim(b \to \sim c) \odot \sim[b \odot \sim(b \to \sim c) \to \sim a] = b \odot \sim(b \to \sim c) \odot \sim[\sim(b \to \sim c) \to (b \to \sim a)] = b \odot [\sim(b \to \sim c) \otimes \sim(b \to \sim a)]$; the last term is symmetric in a and c; the associativity of $\otimes$ and then also of $\oplus$ follows. So (MV2) is proved.

It is not difficult to see that $\otimes$ is isoton, which implies (MV3). (MV4) holds by (LP4). We next prove $a \wedge_L b = a \wedge b$, where $a \wedge_L b = a \otimes (\sim a \oplus b)$; it then follows (MV5). By (11), $a \wedge_L b = a \odot \sim(a \to a \odot \sim(a \to b)) = a \odot \sim(\sim(a \to b) \vee \neg a) = a \odot ((a \to b) \wedge \sim\neg a) = a \odot (a \to b) = a \wedge b$, because $a \odot \sim\neg a = a \odot \neg\neg a = a \odot (\neg a \vee \neg\neg a) = a$ by (LP4)(b) and (LP3)(a).

We next see that $(L; \leq, \oplus, \odot, \sim, 0, 1)$ is a PŁ′-algebra. (PL1) is proved, (PL2) holds by (LP1). (PL3) follows by Lemma 3.2 from $a \odot (b \ominus c) = a \odot b \ominus a \odot c$.

To see the latter, note that (11) implies $a \odot b \odot \sim(b \to c) = a \odot b \odot \sim[(b \to c) \vee \neg a] = a \odot b \odot \sim(a \odot b \to a \odot c)$, because $a \odot \sim\neg a = a$. To see (PL4), apply again Lemma 5.3: $a \odot a = 0$ means $1 = a \odot a \to 0 = (a \to 0) \vee \neg a = \neg a$, so $a = 0$.

We next prove that $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ is a PŁ$'_\triangle$-algebra, where $\triangle = \neg\sim$. By (LP3) and (LP4), $\triangle a \leq \sim\sim a = a$, $\triangle\triangle a = \neg\neg\neg\sim a = \neg\sim a = \triangle a$, $\triangle 1 = \neg 0 = 1$, $\triangle a \vee \sim\triangle a = \neg\sim a \vee \neg\neg\sim a = \neg(\sim a \wedge \neg\sim a) = 1$, and $\triangle(a \vee b) = \triangle a \vee \triangle b$; so (PLb)(a)–(e) are proved.

Note further that $\triangle(a \to b) = \triangle(a \Rightarrow b)$. Indeed, we have $\neg a \leq a \to b$, so $\neg a = \triangle\neg a \leq \triangle(a \to b)$; so, using Lemma 5.3, $\triangle(a \Rightarrow b) = \neg[a \odot \sim(a \to b)] = \neg\sim(a \to b) \vee \neg a = \triangle(a \to b)$.

We may now conclude $\triangle(a \Rightarrow b) \otimes \triangle a = \triangle(\sim b \Rightarrow \sim a) \otimes \triangle a = \triangle(\sim b \to \sim a) \otimes \triangle a \leq \triangle(\sim b \to \sim a) \odot \triangle a \leq (\sim b \to \sim a) \odot (\sim a \to 0) \leq \neg\sim b = \triangle b$; so by (3), (8) follows, and (PLb)(f) is proved.

Divisibility of L holds by (LP2)(a), and equation (14) holds by (LP1). So the first part of the theorem is proved. □

The proof of the second part of Theorem 5.4 follows below.

We next insert the statement that our notion of an ŁΠ-algebra coincides with the usual one.

Proposition 5.5 *ŁΠ-algebras are the structures* $(L; \leq, \odot, \to, \sim, 0, 1)$ *such that:*

(i) $(L; \leq, \odot, \to, 0, 1)$ *is a product algebra,*

(ii) $(L; \leq, \oplus, \sim, 0, 1)$ *is an MV-algebra, where* $\oplus$ *is defined by* (12) *and* (10)*,*

(iii) (7) *holds, where* $\ominus$ *is given by* (1)*.*

Proof. If L is an ŁΠ-algebra, (i) holds by (LP1)-(LP3) and Lemma 5.3. (ii) holds by the first part of Theorem 5.4, which also implies (iii).

For the converse, only (LP4)(a),(b) have to be proved. We refer to [Cin, Corollary 2]. □

Remark 5.6 We have the following relationships of our ŁΠ-algebra axioms to other ones.

- The picture in [Cin] is reflected by Proposition 5.5: an ŁΠ-algebra contains both an MV-algebra and a product algebra in a way that $\odot$ distributes over $\ominus$. Comparing these facts with Definition 5.1, we find that (LP4)(a),(b) is newly added. On the other hand, the cancellation axiom for product algebras is weakened; the axioms for MV-algebras are reduced to the one expressing commutativity of the Łukasiewicz conjunction; and the distributivity axiom is dropped.

- Our results imply that ŁΠ-algebras are $\Pi_\sim$-algebras in which the (derived) Łukasiewicz conjunction is commutative. $\Pi_\sim$-algebras are the algebras $(L; \leq, \odot, \to, \sim, 0, 1)$ corresponding to the equally named logic [EGHN]. They fulfil (LP1)–(LP4), but not necessarily (LP5). On the other hand, every ŁΠ-algebra is a $\Pi_\sim$-algebra. So: Postulating (LP5) for $\Pi_\sim$-algebras leads exactly to the ŁΠ-algebras.

We now proceed with our original line of reasoning. Next, we characterise those product effect algebras which correspond to the ŁΠ-algebras. We naturally repeat Definition 5.2 for $\text{PŁ}'_\triangle$-algebras.

Definition 5.7 We call an f-product effect algebra $(L; \leq, +, \odot, 0, 1)$ *divisible* if for any a, b such that $a \leq b$, there is some k such that $b \odot k = a$.

Theorem 5.8 *Let* $(L; \leq, \odot, \to, \sim, 0, 1)$ *be an* ŁΠ*-algebra. Let* $\oplus$ *be defined according to* (12)*, and let* $+$ *be the partial addition belonging to* $\oplus$*. Then* $(L; \leq, +, \odot, 0, 1)$ *is an f-product effect algebra which is torsion-free and divisible and fulfils strict comparability.* $\to$ *is reobtained by* (14)*, and* $\sim a = 1 - a$ *for any* a*.*

Every f-product effect algebra which is torsion-free and divisible and fulfils strict comparability arises in this way from a unique ŁΠ*-algebra.*

Proof. Let $(L; \leq, \odot, \to, \sim, 0, 1)$ be an ŁΠ-algebra. Then, by the first part of Theorem 5.4, $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ is a divisible $\text{PŁ}'_\triangle$-algebra, $\to$ is given by (14), and $\sim a = 1 - a$ holds by Theorem 2.5. Furthermore, by Theorem 4.8, $(L; \leq, +, \odot, 0, 1)$ is a torsion-free f-product effect algebra fulfilling strict comparability, and L is then also divisible. This completes the proof of the first part.

Let now $(L; \leq, +, \odot, 0, 1)$ be a torsion-free, divisible f-product effect algebra fulfilling strict comparability. $\sim\colon\ L \to L,\ \ a \mapsto 1 - a$ is the usual complementation of effect algebras, thus an involutive, order-reversing operation;

this is the first statement of (LP4). We have to show that $\rightarrow$ exists according to (14); then, taking into account that $\odot$ is by Lemma 3.5(i) isotone, (LP1) will be proved. So let $a, b \in L$. Let $e = e_a^b$ be the sharp element such that $a \wedge e \leq b \wedge e$ and $a \wedge f > b \wedge f$ for any non-zero sharp $f \leq \sim e$.

We claim that there is exactly one $y \leq \sim e$ such that $(a \wedge \sim e) \odot y = b \wedge \sim e$, which by Lemmas 3.5(ii) and 4.4(iii) actually means $a \odot y = b \wedge \sim e$. By divisibility, there is at least one such y. If there are two distinct ones, they may, by Lemma 3.5(ii), be assumed comparable. Let $y' \leq y$ fulfil the condition as well. Then $a \odot (y - y') = 0$, so, because L is torsion-free, we have by Lemma 3.5(iv) $a \wedge (y - y') = 0$; we further conclude by Lemma 4.7(iii) $a \wedge \overline{y - y'} = 0$, where $\overline{y - y'} \leq \sim e$ is sharp; so by assumption, $y = y'$.

We are now able to show that to set $a \rightarrow b = y \vee e$ is consistent with (14). Indeed, we have on the one hand $a \odot (y \vee e) = (a \odot y) \vee (a \odot e) \leq (b \wedge \sim e) \vee (b \wedge e) = b$. On the other hand, let $x \in L$ be such that $a \odot x \leq b$. Then $a \odot (x \wedge \sim e) = (a \odot x) \wedge (a \odot \sim e) \leq b \wedge \sim e$, and consequently, $a \odot [(x \wedge \sim e) \vee y] = [a \odot (x \wedge \sim e)] \vee (a \odot y) = b \wedge \sim e$. From the unicity of y, we conclude $x \wedge \sim e \leq y$ and $x \leq (x \wedge \sim e) \vee e \leq y \vee e$. This completes the proof that the maximum in (14) exists.

If here $b = 0$, we have $a \wedge e = 0$ and $a \odot y = 0$. It follows $a \wedge y = 0$ by Lemma 3.5(iv), hence $a \wedge \neg a = a \wedge (y \vee e) = 0$ in this case. So also (LP3)(a) is shown.

It also follows $(a \rightarrow b) \vee (b \rightarrow a) = 1$, that is (LP2)(b). Indeed, if we repeat the above construction with a and b interchanged, we get $b \rightarrow a = y' \vee e'$, where $e \vee e' = 1$ by Lemma 4.7(i).

(LP2)(a) holds by the divisibility of L. We conclude that L is an BL-algebra.

From $a \wedge \neg a = 0$, we conclude that $a + \neg a$ exists, so $\neg a \leq \sim a$. This proves (LP4)(a). Furthermore, we have $\neg a = \neg a \odot (\neg a \vee \neg\neg a) = \neg a \odot \neg a$, so $\neg a \odot \sim\neg a = \neg a \odot (1 - \neg a) = \neg a - \neg a \odot \neg a = 0$, which means by residuation $\sim\neg a \leq \neg\neg a$. This together with (LP4)(a) proves (LP4)(b). Note in particular that $\neg a$, $a \in L$, are exactly the sharp elements of L.

To see (LP3)(b), assume $\neg\neg a = 1$ and $a \odot b = a \odot c$. We may assume $b \leq c$. From $a \odot (c - b) = 0$, it follows $a \wedge (c - b) = 0$ by Lemma 3.5(iv) and $\overline{a} \wedge (\overline{c - b}) = 0$ by Lemma 4.7(iii). Furthermore, $a \leq \overline{a}$ implies $\neg\neg a \leq \neg\neg\overline{a} = \overline{a}$ because $\overline{a}$ is sharp, and it follows $b = c$.

Finally, let $\otimes'$ be the multiplication of the MV-algebra belonging to the MV-effect algebra $(L; \leq, +, 0, 1)$ according to Theorem 2.5; then $a \otimes' b = \sim(\sim a + (a \wedge \sim b)) = a - (a \wedge \sim b)$. We have $\otimes = \otimes'$ because $a \otimes b = a \odot$

$\sim(a \to \sim b) = a \odot (1 - (a \to \sim b)) = a - a \odot (a \to \sim b) = a - (a \wedge \sim b) = a \otimes' b$, so (LP5) is proved. □

Proof (of the second part of Theorem 5.4). Let $(L; \leq, \oplus, \odot, \sim, 0, 1, \triangle)$ be a divisible $\text{PŁ}'_\triangle$-algebra. Then, $+$ being the natural partial addition, $(L; \leq, +, \odot, 0, 1)$ is, by Theorem 4.8, a torsion-free, divisible f-product effect algebra fulfilling strict comparability. Moreover, $\oplus$ is determined by (5), $\sim$ is determined by (6), and $\triangle a = \sim \overline{\sim a}$ for any a.

By Theorem 5.8, we may define the function $\to$ according to (14) to get an ŁΠ-algebra $(L; \leq, \odot, \to, \sim, 0, 1)$. Like in the proof of Theorem 5.8, we see that $\oplus$ defined by (12) and (10) coincides with $\oplus$ defined by (5). Furthermore, the smallest sharp element majorizing some a is $\neg\neg a$; it follows $\triangle a = \sim \overline{\sim a} = \neg \sim a$ in accordance with (13). □

We finally consider the f-rings representing ŁΠ-algebras. The (algebraic) characterisation of these rings is difficult. In [Mon1], a further operation is postulated. We proceed here in the following, pragmatic way, which is inspired by [Mon1].

Definition 5.9 Let $(G; \leq, +, \odot, 0, 1)$ be an f-ring. Let Div G contain all $g \in G^+$ with the following property: for all $h \in G^+$ such that $h \leq g$ there is a $h_0 \in G^+$ such that $2h_0 = h_0 + h_0 = h$. We call G *weakly divisible* if for all $a, b \in \text{Div } G$ such that $a \leq b$ there is a $k \in G^+$ such that $a = b \odot k$.

Theorem 5.10 *Let $(L; \leq, +, \odot, 0, 1)$ be an f-product effect algebra which is torsion-free and fulfils strict comparability; and let $(\mathcal{R}(L); \leq, +, \odot, 0, 1)$ be the representing f-ring. Then L is divisible if and only if $\mathcal{R}(L)$ is weakly divisible.*

Proof. Let L be divisible, and let $a, b \in \text{Div } \mathcal{R}(L)$ such that $a \leq b$. Because $\mathcal{R}(L)$ is, as an abelian ℓ-group, unperforated, there is a power of 2, say n, which is large enough such that there are $a', b' \in L$ fulfilling $a = na'$ and $b = nb'$. Then $a' = b' \odot k$ for some $k \in L$, and we get $a = b \odot k$ as well. So $\mathcal{R}(L)$ is weakly divisible.

Conversely, let $\mathcal{R}(L)$ be weakly divisible. The rest of the proof refers to the unit interval of $\mathcal{R}(L)$, that is, to L, only. Moreover, in view of Theorem 4.8, we will use the operations with which L is endowed as a $\text{PŁ}'_\triangle$-algebra.

Note first that any a (that is, any $a \in L$) may be composed into $a = a_s + a_d$, where a_s is a sharp element, $a_s \wedge a_d = 0$, and for any $c \leq a_d$, $c = 2c_0$ for some c_0. Indeed, set $a_s = \triangle a = \neg{\sim}a$, and $a_d = a \wedge {\sim}a_s$. Then a_s is sharp, and $a = a_s \vee a_d = a_s + a_d$, because $a_s \wedge a_d = 0$.

Let now $z = a_d \wedge {\sim}a_d = a \wedge {\sim}a$; then $2z$ exists, and $\neg\neg(a_s \vee 2z \vee \neg a) \geq \neg\neg(a_s \vee z \vee \neg a) = 1$. Furthermore, let $h = 2z \to z$; then $2z \odot h = z$. We have $h \wedge \neg\neg z = {\sim}h \wedge \neg\neg z$; this follows by (LP3)(b) from $(a_s \vee 2z \vee \neg a) \odot (h \wedge \neg\neg z) = 2z \odot (h \wedge \neg\neg z) = (2z \odot h) \wedge (2z \odot \neg\neg z) = z \wedge 2z = z$ as well as $(a_s \vee 2z \vee \neg a) \odot ({\sim}h \wedge \neg\neg z) = 2z \odot ({\sim}h \wedge \neg\neg z) = (2z \odot {\sim}h) \wedge (2z \odot \neg\neg z) = (2z \odot (1-h)) \wedge 2z = z \wedge 2z = z$. So we have for any $c \leq a_d$, $c = c \odot \neg\neg z = c \odot (h + {\sim}h) \odot \neg\neg z = c \odot 2(h \odot \neg\neg z) = 2(c \odot h \odot \neg\neg z) = 2(c \odot h)$.

Given now $a, b \in L$ such that $a \leq b$, we decompose $a = a_s + a_d$ and $b = b_s + b_d$ as explained above. Then $a_s \leq b_s$; both $a_d \odot {\sim}b_s$ and b_d belong to Div $\mathcal{R}(L)$; and $a_d \odot {\sim}b_s = a \odot {\sim}b_s \leq b \odot {\sim}b_s = b_d$. It follows $a_d \odot {\sim}b_s = b_d \odot k$ for some $k \leq {\sim}b_s$, and so $a = a \odot (b_s + {\sim}b_s) = a \odot b_s + a_d \odot {\sim}b_s = (a \odot b_s) \odot b_s + k \odot b_d = (a \odot b_s + k) \odot b$. The proof that L is weakly divisible is complete. □

Again, we may conclude the section stating that there is a one-to-one correspondence between ŁΠ-algebras, divisible, torsion-free f-product algebras with strict comparability, and weakly divisible, torsion-free unital f-rings with strict comparability.

6 Summary

We may summarize the contents of this article as follows. In the horizental direction, we have one-to-one correspondences; in the vertical direction, generality decreases.

total algebras	partial algebras	po-groups or po-rings
MV-algebras	lattice-ordered effect algebras with Riesz interpolation	unital abelian ℓ-groups
PŁ-algebras	f-product effect algebras	unital f-rings
PŁ′-algebras	torsion-free f-product effect algebras	unital torsion-free f-rings
$PŁ'_{\triangle}$-algebras	torsion-free f-product effect algebras with strict comparability	unital torsion-free f-rings with strict comparability
ŁΠ-algebras	f-product effect algebras which are divisible, torsion-free, and with strict comparability	unital f-rings which are weakly divisible, torsion-free, and with strict comparability

So it becomes clear that we discussed a chain of five algebras belonging to logics each of which contains the preceeding one. We conclude by noting that there are lots of more logics and more algebras appearing in literature which could be, or have been, studied in an analogous way.

For instance, as mentioned, in [HoCi] still another logic is defined, namely $PŁ_{\triangle}$, which is PŁ enriched by the $\triangle$-connective. Like $PŁ'_{\triangle}$, $PŁ_{\triangle}$ lies between PŁ- and ŁΠ-logics, but it is not comparable to PŁ′. The corresponding algebras are the $PŁ_{\triangle}$-algebras [HoCi].

As a second example, recall that for the weak divisibility of f-rings (Definition 5.9), we referred to the property that certain positive elements are multiples of smaller ones. Such a condition has also been considered in connection with an extension of Łukasiewicz logic [Ger]. On the algebraic side, this condition makes sense even for effect algebras [Pul].

Acknowledgements

This research was supported by the German Science Foundation (DFG) as part of the Collaborative Research Center "Computational Intelligence" (SFB 531, subproject A1).

References

[Baa] M. Baaz, Infinite-valued Gödel logics with 0-1-projections and relativizations, in: P. Hájek (ed.), "Gödel '96. Logical foundations of mathematics, computer science and physics – Kurt Gödel's legacy", Springer-Verlag., Berlin 1996, pp. 23 - 33.

[Bir] G. Birkhoff, "Lattice Theory", AMS Coll. Publ., Rhode Island 1995 (3-rd edition).

[ChKo] F. Chovanec, F. Kôpka, Boolean D-posets, *Tatra Mt. Math. Publ.* **10** (1997), 183 - 197.

[Cin] P. Cintula, A note on the definition of the ŁΠ-algebras, *Soft Comp.*, to appear.

[CiOtMu] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, "Algebraic Foundations of Many-valued Reasoning", Kluwer Academic Publ., Dordrecht, 2000.

[DiDv] A. Di Nola, A. Dvurečenskij, Product MV-algebras, *Mult.-Valued Log.* **6** (2001), 193 - 215.

[Dvu] A. Dvurečenskij, Product effect algebras, *Int. J. Theor. Phys.* **41** (2002), 1827 - 1839.

[DvPu] A. Dvurečenskij, S. Pulmannová, "New trends in quantum structures", Kluwer Academic Publ., Dordrecht, and Ister Science, Bratislava 2000.

[EsGo] F. Esteva, L. Godo, Putting together Łukasiewicz and product logics, *Mathware Soft Comput.* **6** (1999), 219 - 234.

[EGHN] F. Esteva, L. Godo, P. Hájek, M. Navara, Residuated fuzzy logics with an involutive negation, *Arch. Math. Logic* **39** (2000), 103 - 124.

[EsGoMo] F. Esteva, L. Godo, F. Montagna, The $L\Pi$ and $L\Pi\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics, *Arch. Math. Logic* **40** (2001), 39 - 67.

[FoBe] D. J. Foulis, M. K. Bennett, Effect algebras and unsharp quantum logics, *Found. Phys.* **24** (1994), 1325 - 1346.

[Fuc] L. Fuchs, "Partially Ordered Algebraic Systems", Pergamon Press, Oxford, 1963.

[Ger] B. Gerla, Rational Łukasiewicz logic and DMV-algebras, *Neural Network World* **11** (2001), 579 - 594.

[Goo] K. R. Goodearl, "Partially Ordered Abelian Groups with Interpolation", American Mathematical Society, Providence, 1986.

[GrFoPu] R. J. Greechie, D. Foulis, S. Pulmannová, The center of an effect algebra, *Order* **12** (1995), 91 - 106.

[Haj] P. Hájek, "Metamathematics of Fuzzy Logic", Kluwer Acad. Publ., Dordrecht 1998.

[HaGoEs] P. Hájek, L. Godo, F. Esteva, A complete many-valued logic with product-conjunction, *Arch. Math. Logic* **35** (1996) 191 - 208.

[HoCi] R. Horčík, P. Cintula, Product Łukasiewicz logic, *Arch. Math. Logic* **43** (2004), 477 - 503.

[MaMu] V. Marra, D. Mundici, Łukasiewicz logic and Chang's MV-algebras in action, in: V. F. Hendricks et al. (eds.), "Trends in logic. 50 years of Studia Logica", Kluwer Academic Publ., Dordrecht 2003, pp. 145 - 192.

[Mon1] F. Montagna, An algebraic approach to propositional fuzzy logic, *J. Logic Lang. Inf.* **9** (2000), 91 - 124.

[Mon2] F. Montagna, Functorial representation theorems for MV_Δ algebras with additional operators, *J. Algebra* **238** (2001), 99 - 125.

[Mon3] F. Montagna, Subreducts of MV-algebras with product and product residuation, to appear.

[Mun] D. Mundici, Interpretation of AF $C^\star$-algebras in Łukasiewicz sentential calculus, *J. Funct. Anal.* **65** (1986), 15 - 63.

[MoPa] F. Montagna, G. Panti, Adding structure to MV-algebras, *J. Pure Appl. Algebra* **164** (2001), 365 - 387.

[Pul] S. Pulmannová, Divisible effect algebras and interval effect algebras, *Comment. Math. Univ. Carolinae* **42** (2001), 219 - 236.

[Rav] K. Ravindran, On a structure theory of effect algebras, Ph.D. Thesis, Kansas State University, Manhattan 1996.

[Vet] T. Vetterlein, BL-algebras and effect algebras, *Soft Comp.*, to appear.