Analytic calculi for logics of ordinal multiples of standard t-norms

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Abstract

For two propositional fuzzy logics, we present analytic proof calculi, based on relational hypersequents. The logic considered first, called **MŁ**, is based on the finite ordinal sums of Łukasiewicz t-norms. In addition to the usual connectives – the conjunction $\odot$, the implication $\to$, and the constant 0 –, we use a further unary connective interpreted by the function associating with each truth value a the greatest $\odot$-idempotent below a. **MŁ** is a conservative extension of Basic Logic.

The second logic, called **MΠ**, is based on the finite ordinal sums of the product t-norm on $(0, 1]$. Our connectives are in this case just the conjunction and the implication.

1 Introduction

In recent years, the proof theory of fuzzy logics has developed considerably. Let us shortly review the present situation on the basis of the most important examples.

By a fuzzy logic, we mean any many-valued propositional logic whose propositions are interpreted in the real unit interval and whose language contains a conjunction and an implication interpreted by the adjoint pair of a t-norm and its residuum. The logic **MTL** [9] is the most general such logic, requiring not more than left-continuity of the t-norm. Basic Logic [12], or **BL** for short, is the logic of all continuous t-norms. Finally, there are the logics based on one of the three standard t-norms: Łukasiewicz, product, and Gödel logic [8, 12].

For the latter three logics, proof systems with convenient proof-theoretical properties were found [15, 16, 2]. Most important, all these system enjoy the subformula property, and a proposition to be proved can be decomposed step by step into its constituents. The principal tool used are hypersequents, or a variant hereof called r- (i.e. relational) hypersequents. In the comprehensive paper [7], all these logics are even treated in a uniform way, the rules introducing the binary logical connectives being invertible. Moreover, an analytic hypersequent calculus was found for **MTL** [4]. In this case, completeness was shown for a system with a cut rule, and the redundancy of the cut rule was proved in the Schütte-Tait style. What remains, however, is the well-known logic **BL**. Strangely enough, it seems to be difficult if not impossible to formulate a proof system for **BL** along the same lines. So modifications of the concept seem unavoidable. One successful attempt in this direction was recently made by S. Bova and F. Montagna; in [5], the notion of a relational hypersequent was non-trivially generalized, and invertible logical rules were defined. The validity of atomic hypersequents, though, is to be checked by linear programming methods.

A further contribution to the proof theory of **BL** is contained in the first half of the present paper. We offer a kind of indirect solution of the problem how to define an analytic hypersequent calculus for **BL**. Namely, we consider a fuzzy logic based on finite ordinal multiples of the Łukasiewicz t-norm. As long as the conjunction, implication and the constant 0 are involved, this is **BL**; the ordinal sums of Łukasiewicz t-norms generate already the variety of BL-algebras [18]. Here, however, we will enrich the language by a unary connective $\triangledown$, interpreted by the function which maps a truth value to the greatest $\odot$-idempotent below it. This connective was studied for **BL** by Hájek [13] and for arbitrary many-valued logics by Montagna [17]. However, our aim is different. Recall that any continuous t-norm is build up, using the ordinal sum construction, from the Łukasiewicz, the product, and the Gödel t-norm. Our motivation to introduce the additional connective $\triangledown$ is to exclude all those continuous t-norms in whose construction the product t-norm is involved.

We call the new logic **MŁ**, and as it is easily seen, **MŁ** is a conservative extension of **BL**. Now, for **MŁ**, the problem of finding invertible hypersequent rules turns out to be much easier than for **BL**. We shall propose an r-hypersequent calculus for **MŁ** called **rHMŁ**. This calculus is still somewhat more involved than any of those known for other kinds of fuzzy logics. The reason is that the new connective $\triangledown$ requires special treatment; there are no invertible rules for $\triangledown$ itself. We rather have to formulate two separate rules for each binary connective and each side, depending on the appearance of $\triangledown$ as the outermost connective. Furthermore, also the remaining rules become more involved by the presence of $\triangledown$.

Our calculus **rHMŁ** fulfills the subformula property in a restricted sense. Namely, for any formula α appearing in one of the assumptions, either α is the subformula of a formula in the conclusion; or else α is of the form $\triangledown\beta$, and β is the subformula of a formula in the conclusion. Furthermore, the rules concerning the logical connectives $\odot$ and $\rightarrow$ are invertible. The remaining rules are applied to r-hypersequents consisting of expressions of the form α or $\triangledown\alpha$, where α is an atom. In this part, a version of the cut rule appears on the external level; it is what might be called an "analytic" cut, distinguishing the cases $\triangledown\alpha \leq \triangledown\beta$ and $\triangledown\beta < \triangledown\alpha$, where α and β are atoms contained in the proposition to be proved. Cf. the logical rules in [5].

In the second half of this paper, we consider a further logic called **MΠ**. Whereas **MŁ** is based on ordinal sums of an algebra generating the variety of bounded Wajsberg hoops, **MΠ**, in contrast, is based on ordinal sums of an algebra generating the variety of unbounded Wajsberg hoops [10]. Namely, **MΠ** is based on ordinal multiples of the standard cancellative hoop [10], whose monoidal operation is $(0,1]^2 \rightarrow (0,1], \;\; (a,b) \mapsto a \cdot b$. We have to use a proper subset of the real unit interval, in particular not containing 0, for the truth values, and we work with the conjunction $\odot$ and the implication $\rightarrow$ as the only connectives. Note that to include the constant 0 and to work with the closed real unit interval would result in a logic based on ordinal sums of unbounded Wajsberg hoops with singletons inserted, not well in accordance with our concept.

The rules of our calculus **rHMΠ** introducing the binary connectives are similar to those of **rHMŁ**. However, the remaining rules of **MΠ** are not chosen in analogy to **rHMŁ**; they are rather based on a different concept.

2 The logics MŁ and MΠ

Subject of this paper are two t-norm based many-valued logics. We begin with their model-theoretic definition.

A *t-norm* is a binary operation $\odot$ on a subset L of the real unit interval $[0,1]$ containing 1 such that $(L; \leq, \odot, 1)$ is an ordered monoid, $\leq$ being the natural order. Provided that a residual function $\to$ corresponding to $\odot$ exists, we call $(L; \odot, \to, 1)$ the *t-norm algebra* determined by $\odot$.

A particularly simple algebraic structure is characteristic for the Łukasiewicz t-norm, defined by $\odot_{\mathrm{L}}: \ [0,1]^2 \to [0,1], \ \ (a,b) \mapsto (a+b-1) \vee 0$, and for the product t-norm restricted to the half-open unit interval, given by $\odot_\Pi: \ (0,1]^2 \to (0,1], \ \ (a,b) \mapsto a \cdot b$. (Here, like in the sequel, $\wedge$ and $\vee$ denote the infimum and the supremum, respectively, w.r.t. the natural order of the reals.) The t-norm algebras determined by $\odot_{\mathrm{L}}$ and $\odot_\Pi$ are called *Łukasiewicz algebra* and *standard cancellative hoop*, respectively.

Furthermore, we may construct new t-norm algebras out of given ones by means of ordinal summation. We will understand this well-known construction in accordance with [1, Section 3]; in particular, the 1 elements of all components are identified. In this paper, we will be concerned with t-norm algebras which are the ordinal sum of finitely many isomorphic copies of one and the same algebra – namely either the Łukasiewicz algebra or the standard cancellative hoop.

Definition 2.1 The *Logic of Multiple Łukasiewicz t-Norms*, or **MŁ** for short, is a propositional logic with the binary connectives $\odot$ and $\to$, the constant 0, and the unary connective $\triangledown$. The set of propositions of **MŁ** is denoted by $\mathcal{P}_{\mathbf{M}\mathbf{Ł}}$. An evaluation of **MŁ** is a mapping from $\mathcal{P}_{\mathbf{M}\mathbf{Ł}}$ to an algebra $([0,1]; \odot, \to, 0, \triangledown)$ preserving the connectives and the constant, where (i) $([0,1]; \odot, \to, 1)$ is a t-norm algebra such that, for a sequence $0 = a_0 < a_1 < \ldots < a_k = 1$, each subset $[a_{i-1}, a_i) \cup \{1\}$, $i = 1, ..., k$, determines a subalgebra which is isomorphic to the Łukasiewicz algebra, and (ii)

$$\triangledown\colon \ [0,1] \to [0,1], \ \ a \mapsto \ \max\ \{x \in [0,1]\colon \ x \leq a, \ x \odot x = x\}. \qquad (1)$$

The *valid* propositions of **MŁ** are those being assigned 1 by all evaluations.

Let us first comment the fragment of **MŁ** which contains only the connectives $\odot$, $\to$, and 0. For detailed information on the Basic (Fuzzy) Logic **BL**, the logic of all continuous t-norms, we refer to Hájek's monograph [12].

Theorem 2.2 *Let α be a proposition of* **MŁ** *not containing the connective $\triangledown$. Then α is valid in* **MŁ** *if and only if α is valid in* **BL**.

Proof. It is well-known that α is valid in **BL** exactly if α is valid in all t-norm algebras $([0,1]; \odot, \to, 1)$ which are a finite ordinal sum of Łukasiewicz algebras; see e.g. [18]. □

In particular, **MŁ** is a conservative extension of **BL**.

MŁ is closely related to a logic which was considered earlier by several authors. Namely, Hájek introduced in [13] the extension BL_{lu} of **BL**, whose language comprises an additional unary connective interpreted in exact accordance with the formula (1). In contrast to the present case, however, Hájek allows the conjunction $\odot$ to be an arbitrary continuous t-norm.

It follows that the fragment of Hájek's BL_{lu} arising from the restriction to those formulas containing only $\odot, \rightarrow, 0, \triangledown$, is strictly weaker than **MŁ**. The proposition $(((\alpha \rightarrow \triangledown\alpha) \rightarrow \triangledown\alpha)) \rightarrow \alpha$, for instance, is valid in **MŁ**, but not in BL_{lu}.

We further note that the ideas of [13] were further developed in the paper [17]. In [17], the extension of an arbitrary many-valued logic by what is called a storage operator is studied.

We now turn to the second logic which we will study: **MΠ**. **MΠ** is based on ordinal sums of the standard cancellative hoop. This implies that we cannot choose the whole real unit interval as our set of truth values; we have to restrict to the union of finitely many open subintervals plus the 1 element.

Definition 2.3 The *Logic of Multiple Product t-Norms*, or **MΠ** for short, is a propositional logic with the binary connectives $\odot$ and $\rightarrow$. The set of propositions of **MΠ** is denoted by $\mathcal{P}_{\mathbf{M\Pi}}$. An evaluation of **MΠ** is a mapping from $\mathcal{P}_{\mathbf{MŁ}}$ to an algebra $(L; \odot, \rightarrow)$ preserving the connectives, where $(L; \odot, \rightarrow, 1)$ is a t-norm algebra such that, for a sequence $0 = a_0 < a_1 < \ldots < a_k = 1$, we have $L = \bigcup_{1 \leq i \leq k} (a_{i-1}, a_i) \cup \{1\}$ and each subset $(a_{i-1}, a_i) \cup \{1\}$, $i = 1, ..., k$, determines a subalgebra which is isomorphic to the standard cancellative hoop. The *valid* propositions of **MΠ** are those being assigned 1 under all evaluations of **MΠ**.

The reason to consider the logics **MΠ** and **MŁ** together in one paper is that they may be opposed to each other quite naturally. Namely, **MŁ** can be seen as the logic of ordinal sums of linearly ordered bounded Wajsberg hoops, which in turn are representable by means of intervals of the negative cone of Abelian linearly ordered groups. As opposed to that, **MΠ** is the logic of ordinal sums of linearly ordered *un*bounded Wajsberg hoops, which in turn are representable by means of the whole negative cone of Abelian linearly ordered groups. For these facts, cf. e.g. [10, 1]. – Note that **MΠ** may also be described as the logic of linearly ordered basic hoops fulfilling the equation $a \rightarrow (a \odot a) = a$.

Finally, let us introduce explicitely also the Łukasiewicz logic **Ł** [8], to which we will have to refer at certain points. A proposition in the language $\odot, \rightarrow, 0$ of **Ł** is called a valid proposition of **Ł** if it is assigned 1 by all evaluations in $([0,1]; \odot, \rightarrow, 0)$ such that $([0,1]; \odot, \rightarrow, 1)$ is the Łukasiewicz algebra.

3 Algebraic preliminaries

Following the common practice for fuzzy logics, also the semantics of **MŁ** and **MΠ** is based on the real unit interval or a subset of it. However, to write down the explicit expression of an interpretation of the connectives $\odot$ and $\rightarrow$ is rather intricate. For this reason, we will not work in the sequel with the canonical models for **MŁ** and **MΠ** given in Definitions 2.1 and 2.3, but with isomorphic copies which are easier to handle.

Note first that the Łukasiewicz algebra is isomorphic to the algebra $([-1,0], \odot, \rightarrow, z, e)$, where $[-1,0] = \{r \in \mathbb{R} \colon -1 \leq r \leq 0\}$ is an interval of the negative cone of the linearly ordered group of real numbers, the operations $\odot$ and $\to$ are given by

$$\begin{aligned} r \odot s &= (r+s) \vee -1, \\ r \to s &= \begin{cases} s-r & \text{if } r > s \\ 0 & \text{else,} \end{cases} \end{aligned}$$

for $r, s \in [-1, 0]$, and $z = -1$, $e = 0$. Similarly, the standard cancellative hoop is isomorphic to $(\mathbb{R}^-; \cdot, \to, e)$, where $\mathbb{R}^- = \{r \in \mathbb{R} : r \leq 0\}$ is the negative cone of the linearly ordered group of reals, furthermore

$$\begin{aligned} r \cdot s &= r + s, \\ r \to s &= \begin{cases} s-r & \text{if } r > s \\ 0 & \text{else,} \end{cases} \end{aligned}$$

for $r, s \in \mathbb{R}^-$, and $e = 0$. Now, to get models for **MŁ** and **MΠ**, we need to form the ordinal sum of finitely many copies of the former or the latter algebra, respectively.

Definition 3.1 For every natural number $k \geq 1$, we let

$$\begin{aligned} S_k = \{(n, r) \in \mathbb{Z} \times \mathbb{R} :\ & n = 0 \text{ and } -1 \leq r \leq 0, \quad \text{or} \\ & -(k-1) \leq n \leq -1 \text{ and } -1 \leq r < 0\}; \end{aligned}$$

we endow S_k with the lexicographical order; for $(m, r), (n, s) \in S_k$, we define

$$\begin{aligned} (m, r) \odot (n, s) &= \begin{cases} (m,\ (r+s) \vee -1) & \text{if } m = n, \\ (m, r) \wedge (n, s) & \text{else,} \end{cases} \\ (m, r) \to (n, s) &= \begin{cases} (n, s) & \text{if } m > n, \\ (m, s-r) & \text{if } m = n \text{ and } r > s, \\ (0, 0) & \text{if } (m, r) \leq (n, s), \end{cases} \\ \triangledown(n, r) &= \begin{cases} (n, -1) & \text{if } r < 0, \\ (0, 0) & \text{if } (n, r) = (0, 0); \end{cases} \end{aligned}$$

and we let $e = (0, 0)$ and $z_k = (-(k-1), -1)$. Then $(S_k; \leq, \odot, \to, \triangledown, z_k, e)$ will be called the *k-fold Łukasiewicz algebra.*

Moreover, for $k \geq 1$, we let

$$\begin{aligned} T_k = \{(n, r) \in \mathbb{Z} \times \mathbb{R} :\ & n = 0 \text{ and } r \leq 0, \quad \text{or} \\ & -(k-1) \leq n \leq -1 \text{ and } r < 0\}; \end{aligned}$$

we endow T_k with the lexicographical order; for $(m, r), (n, s) \in T_k$, we define

$$\begin{aligned} (m, r) \cdot (n, s) &= \begin{cases} (m,\ r+s) & \text{if } m = n, \\ (m, r) \wedge (n, s) & \text{else,} \end{cases} \\ (m, r) \to (n, s) &= \begin{cases} (n, s) & \text{if } m > n, \\ (m, s-r) & \text{if } m = n \text{ and } r > s, \\ (0, 0) & \text{if } (m, r) \leq (n, s); \end{cases} \end{aligned}$$

and we let $e = (0,0)$. Then $(T_k; \leq, \cdot, \rightarrow, e)$ will be called the *k-fold standard cancellative hoop.*

In the sequel, all evaluations of propositions of **MŁ** will be assumed to be mappings to some S_k, endowed with $\odot$, $\rightarrow$, z_k, $\triangledown$, rather than mappings to $[0,1]$. Similarly, the range of evaluations of **MΠ** will be assumed to be some T_k, endowed with $\odot$ and $\rightarrow$. Validity means to be always assigned the value denoted by e for both logics.

We will now collect some elementary facts about S_k and T_k. Note first that the Archimedean classes of the ordered monoids S_k are $\{(0,0)\}$ and $\{(n,r)\colon -1 \leq r < 0\}$ for $n = 0, -1, \ldots, -(k-1)$; similarly, the Archimedean classes of T_k are $\{(0,0)\}$ and $\{(n,r)\colon r < 0\}$ for $n = 0, -1, \ldots, -(k-1)$. For $a, b \in S_k$ or $a, b \in T_k$, we write $a \sim b$, if a and b are Archimedean equivalent, $a \preccurlyeq b$ if the class of a is equal to or below the class of b, and $a \prec b$ if $a \preccurlyeq b$, but not $a \sim b$. Instead of $a \prec b$, we also write $b \succ a$; instead of $a \preccurlyeq b$, we also write $b \succcurlyeq a$.

In S_k, the relations $\sim, \preccurlyeq, \prec$ are expressible by the operation $\triangledown$:

Lemma 3.2 *Let $k \geq 1$, and let $a, b \in S_k$. Then $a \sim b$ if and only if $\triangledown a = \triangledown b$; $a \preccurlyeq b$ if and only if $\triangledown a \leq \triangledown b$; and $a \prec b$ if and only if $\triangledown a < \triangledown b$.*

In what follows, we will, for any $k = 1, \ldots$, consider S_k as a subset of T_k. Indeed, whereas evaluations of propositions of **MŁ** are done in some S_k, evaluations of multisets of propositions of **MŁ**, which will appear later, will map to T_k.

Note then that for any $a, b \in S_k \subseteq T_k$, it makes a difference if the product is taken in S_k or in T_k; we may only say that $a \cdot b \leq a \odot b$. However, the implication $a \rightarrow b$ is calculated in S_k in the same way as in T_k; for this reason, we do not distinguish these two operations in notation. Moreover, for $a, b \in S_k$, $a \sim b$, $a \preccurlyeq b$, $a \prec b$ holds in S_k exactly if it holds in T_k, respectively.

Lemma 3.3 *Let $k \geq 1$. For any $a, b, c, d \in S_k$, the following holds:*

(i) *$\triangledown a \leq a^n$ for any $n \geq 1$; here, $a^1 = a$ and $a^{k+1} = a^k \odot a$ for $k \geq 1$.*

(ii) *$\triangledown$ is isotone: $a \leq b$ implies $\triangledown a \leq \triangledown b$.*

(iii) *$\triangledown a \sim a$, and $\triangledown a$ is the minimal element in the Archimedean class of a.*

(iv) *Let $a \sim b$. Then $a \odot b \sim a$. If furthermore $a < b$, then $b \rightarrow a \sim a$.*

(v) *$a \preccurlyeq b$ implies $a \odot b \sim a$. Moreover, $a \prec b$ implies $a \odot b = b \rightarrow a = a$.*

Moreover, let $a, b, c, d \in T_k$. Then (iv) *and* (v) *in which $\odot$ is replaced by $\cdot$, hold as well. In addition, we have:*

(vi) *Let $a \preccurlyeq c$. Then $a \leq b$ if and only if $a \cdot c \leq b \cdot c$.*

(vii) *Let $a \sim b$. Then $a = a \cdot b$ implies $a = b = e$.*

We will make use of these facts in the sequel without explicit reference.

Lemma 3.4 *For any $a, b \in S_k$, we have*

$$\nabla(a \odot b) \;=\; \nabla a \wedge \nabla b; \tag{2}$$

$$\nabla(a \to b) \;=\; \begin{cases} e & \text{if } a \leq b, \\ \nabla b & \text{if } a > b. \end{cases} \tag{3}$$

Proof. If $a \leq b$, then $a \odot b \sim a$, whence $\nabla(a \odot b) = \nabla a$. (2) follows.
$a \leq b$ implies $a \to b = e$, and $a > b$ implies $a \to b \sim b$. So also (3) is shown. $\square$

Lemma 3.5 *Let $k \geq 1$, and let $a, b, c, d \in S_k$.*

(i) *The following statements are equivalent:*

(α) $a \cdot b \cdot c \leqq d$, *and either* $\nabla a \cdot c \leqq d$ *or* $\nabla b \cdot c \leqq d$;

(β) $(a \odot b) \cdot c \leqq d$,

where $\leqq$ is uniformly chosen $\leq$ or $<$.

(ii) *The following statements are equivalent:*

(α) $c \leqq d$, *and either* $a \leq b$ *or* $a \cdot c \leqq b \cdot d$;

(β) $c \leqq (a \to b) \cdot d$,

where $\leqq$ is uniformly chosen $\leq$ or $<$.

Proof. (i) Assume first that $a \odot b = a \cdot b$. Then clearly, $a \cdot b \cdot c \leqq d$ iff $(a \odot b) \cdot c \leqq d$. In this case, since $\nabla a \wedge \nabla b \leq a \odot b$, we furthermore have $\nabla a \cdot c \leq (a \odot b) \cdot c \leqq d$ if $\nabla a \leq \nabla b$, and else $\nabla b \cdot c \leqq d$. So (α) iff (β).

Assume now that $a \odot b > a \cdot b$. Then $a \odot b = \nabla a = \nabla b$. So $\nabla a \odot c = \nabla b \odot c \leqq d$ implies $(a \odot b) \cdot c \leqq d$. Conversely, $(a \odot b) \cdot c \leqq d$ implies $a \cdot b \cdot c \leqq d$ and $\nabla a \cdot c \leqq d$. So again, (α) iff (β).

(ii) If $a \leq b$, we have $a \to b = e$, and the equivalence of (α) and (β) is evident. So assume $a > b$. Note that then $a \to b \sim b \preccurlyeq a$.

Let (α) hold, that is, $c \leqq d$ and $a \cdot c \leqq b \cdot d$. Then $a \prec c$ would mean $a = a \cdot c \leq b \cdot d = b$; so $a \not\succcurlyeq c$.

Assume $a \succ c$. If then $a \succ b$, we have $c = a \cdot c \leqq b \cdot d = (a \to b) \cdot d$. If then $a \sim b \succ d$, it follows $c \leqq d = (a \to b) \cdot d$. If finally $a \sim b \preccurlyeq d$, we have $c \prec (a \to b) \cdot d$.

Assume $a \sim c$. Then $a \sim b \sim c \preccurlyeq d$. In case $d \succ c$, we conclude from $a \cdot c \leqq b \cdot d = b = a \cdot (a \to b)$ that $c \leqq a \to b = (a \to b) \cdot d$. In case $d \sim c$, we have $a \cdot c \leqq b \cdot d = a \cdot (a \to b) \cdot d$, so $c \leqq (a \to b) \cdot d$.

So it follows (β) in both cases. For the converse direction, let us assume that (β) holds. Clearly, we then have $c \leqq d$. Moreover, $c \leq a \to b \sim b$. In particular, $c \preccurlyeq d$ and $c \preccurlyeq b \preccurlyeq a$. We have to show $a \cdot c \leqq b \cdot d$.

Let first $a \succ c$. Then $a \cdot c = c \lneqq (a \to b) \cdot d$. If $a \succ b$, then $a \to b = b$, so $a \cdot c \lneqq b \cdot d$. Otherwise, $a \sim b$. If, in this case, $c \prec d$, we have $a \cdot c = c \prec b \cdot d$; if $c \sim d$, then $a \to b \succ d$, so $c = a \cdot c \lneqq d = b \cdot d$.

Let now $a \sim c$. Then $a \sim b \sim c \preccurlyeq d$. If $d \succ b$, we have $c \lneqq a \to b$, and it follows $a \cdot c \lneqq a \cdot (a \to b) = b = b \cdot d$. If $d \sim b$, we similarly conclude that $a \cdot c \lneqq a \cdot (a \to b) \cdot d = b \cdot d$.

So (α) is shown to hold. The proof is complete that (α) and (β) are equivalent. □

Lemma 3.6 *Let $k \geq 1$, and let $a, b, c \in T_k$ such that $a \wedge b \prec c$. Let $\lneqq$ denote one of $\leq$ or $<$. Then $a \lneqq b$ if and only if $a \cdot c \lneqq b$ if and only if $a \lneqq b \cdot c$.*

Proof. Note that it is sufficient to prove the assertion for the case that $\lneqq$ is $\leq$.

We have $a \prec c$ or $b \prec c$. Assume first $a \prec c$. Then $a \cdot c = a$, whence $a \leq b$ iff $a \cdot c \leq b$. Moreover, if $a \leq b$, then $a \leq b = b \cdot c$ in case $a \sim b$, and $a \prec b \cdot c$ in case $a \prec b$. Conversely, $a \leq b \cdot c$ clearly implies $a \leq b$.

Assume next $b \prec c$. $a \leq b$ clearly implies $a \cdot c \leq b$. Conversely, $a \cdot c \leq b$ implies $a \prec c$, so $a \leq b$. Moreover, we evidently have $a \leq b$ iff $a \leq b \cdot c$. □

Lemma 3.7 *Let $k \geq 1$. For any $a, b, c, d \in T_k$, the following holds:*

(i) *$a \leq b$ and $c \leq d$ imply $a \cdot c \leq b \cdot d$.*

(ii) *$a \cdot c \leq b \cdot d$ implies $a \leq b$ or $c \leq d$.*

(iii) *Let $a \wedge b \preccurlyeq c$. Then $a \cdot c \leq b \cdot d$ implies $a \leq b$ or $c < d$.*

Proof. (i) holds by the monotonicity of $\cdot$.

(ii) Assume $a \cdot c \leq b \cdot d$. If $a \prec c$, then $a = a \cdot c \leq b \cdot d \leq b$. If $a \succ c$, then $c = a \cdot c \leq b \cdot d \leq d$.

Let now $a \sim c$. Then $a, c \preccurlyeq b, d$. If $b \prec d$, then $c \prec d$, hence $c \leq d$. Similarly, if $d \prec b$, then $a \prec b$, hence $a \leq b$. Finally, let $b \sim d$. If moreover $a \prec b$, clearly $a \leq b$. Otherwise a, b, c, d are all in the same Archimedean class; $b < a$ and $d < c$ would imply $b \cdot d < a \cdot c$; so $a \leq b$ or $c \leq d$.

(iii) By assumption $a \preccurlyeq c$ or $b \preccurlyeq c$. Let $a \cdot c \leq b \cdot d$ and $d \leq c$. We shall show that then $a \prec c$, $a \sim c$, $b \prec c$, $b \sim c$ all imply $a \leq b$.

Let $a \prec c$. Then $a = a \cdot c \leq b \cdot d \leq b$.

Let $a \sim c$. Then $a \sim c \sim d \preccurlyeq b$. If $a \prec b$, clearly $a \leq b$. Otherwise $a \sim b \sim c$, and $a \cdot c \leq b \cdot d \leq b \cdot c$ implies $a \leq b$.

Let $b \prec c$. From $a \cdot c \leq b$, it then follows $a \prec c$, implying $a \leq b$ as seen above.

Finally, let $b \sim c$. Assume that $a \succ c$. Then $c = a \cdot c \leq b \cdot d \leq b \cdot c \leq c$, that is, $b \cdot c = c$. It follows $b = c = e$, a contradiction. So $a \preccurlyeq c$, implying $a \leq b$ as seen above. □

We conclude this section with a fact about linear orders, needed later in one of the proofs.

Lemma 3.8 *Let $\mathcal{L}$ be the language of bounded posets, that is, let $\mathcal{L}$ contain a binary relation $\leq$, and two constants 0 and 1. Let Φ be a non-empty finite set of $\mathcal{L}$-sentences of the form*

$$a \leq b \quad \text{or} \quad a < b,$$

where a and b are variables or constants, and $a < b$ means $\neg(b \leq a)$. Then the following conditions are equivalent:

(α) *The disjunction $\bigvee \Phi$ holds in all bounded linearly ordered posets $(L; \leq, 0, 1)$.*

(β) *There are variables $a_1, \ldots, a_n$, $n \geq 1$, such that Φ contains one of the following set of statements:*

- (i) *$a_1 \lneqq a_2,\ a_2 \lneqq a_3, \ldots, a_{n-1} \lneqq a_n,\ a_n \leq a_1$, where each symbol $\lneqq$ denotes individually either $\leq$ or $<$, and the case $n = 1$ is understood as $a_1 \leq a_1$;*
- (ii) *$0 < a_1,\ a_1 < a_2, \ldots,\ a_{n-2} < a_{n-1},\ a_{n-1} \leq a_n$, and the case $n = 1$ is understood as $0 \leq a_1$.*
- (ii') *$0 < a_1,\ a_1 < a_2, \ldots,\ a_{n-2} < a_{n-1},\ a_{n-1} \leq 0$, and the case $n = 1$ is understood as $0 \leq 0$.*
- (iii) *$a_1 \leq a_2,\ a_2 < a_3, \ldots,\ a_{n-1} < a_n,\ a_n < 1$, and the case $n = 1$ is understood as $a_1 \leq 1$.*
- (iii') *$1 \leq a_2,\ a_2 < a_3, \ldots,\ a_{n-1} < a_n,\ a_n < 1$, and the case $n = 1$ is understood as $1 \leq 1$.*
- (iv) *$0 < a_2,\ a_2 < a_3, \ldots,\ a_{n-1} < a_n,\ a_n < 1$, and the case $n = 1$ is understood as $0 < 1$.*
- (v) *$0 \leq 1$.*

Proof. Clearly, (β) implies (α). For the converse direction, let Φ be such that (α) holds.

Let us call a subset A of Φ of the form (i)-c if A arises from a set of the form (i) by replacing any of its variables uniformly by one of the two constants. Similarly, we define subsets of the form (ii)-c, (ii')-c, (iii)-c, (iii')-c, and (iv)-c.

In [3], the valid atomic sequents-of-relations in the Gödel logic $\mathbf{R}\mathrm{G}_\infty$ were characterized. It is immediate from [3] that Φ contains a subset Φ_0 which is of the form (i)-c or (ii)-c or (iii)-c or (iv)-c. We have to show that Φ_0 contains a subset of the form (i), (ii), (ii'), (iii), (iii'), or (iv).

The case $n = 1$ is easy; let us assume $n \geq 2$. Also the case that a formula $0 \leq a_i$ or $a_i \leq 1$ is in Φ_0, causes no difficulties; let us assume that formulas of this type are not in Φ_0.

Assume now that Φ_0 is of the form (i)-c and that at least one variable, say a_k, is replaced by a constant. If this constant is 0, then Φ_0 contains a subset of the form (ii)-c or (ii')-c. If this constant is 1, then Φ contains a subset of the form (iii)-c or (iii')-c. It is finally not difficult to see that any set of the form (ii)-c, (ii')-c, (iii)-c, (iii')-c, or (iv)-c contains a subset of the form (ii), (ii'), (iii), (iii'), or (iv). □

4 A hypersequent calculus for MŁ

For the logic **MŁ** of ordinal sums of the Łukasiewicz t-norm algebra, we will present a proof system based on r-hypersequents. The idea to generalize ordinary hypersequents to r-hypersequents, appeared the first time in [7].

All propositions in this section are in the language $\odot, \rightarrow, 0, \triangledown$ of **MŁ**. An *r-sequent* $\Gamma \lesseqqgtr \Delta$ consists of the *antecedent* Γ and the *succedent* Δ, both finite multisets of propositions, and a *relational symbol* $\lesseqqgtr$, which is one of the symbols $\leq$ or $<$. We write $\emptyset$ for the empty multiset. An *r-hypersequent* is a finite multiset of r-sequents, notated by $\Gamma_1 \lesseqqgtr \Delta_1 \mid \ldots \mid \Gamma_k \lesseqqgtr \Delta_k$.

A *rule* is a pair consisting of a finite set of assumptions, i.e. a finite set of r-hypersequents, which is possibly empty and possibly not bounded from above by a fixed number, and a conclusion, i.e. a single r-hypersequent. Rules with no assumptions are called *axioms*. A *proof* is a finite tree-ordered set of substitution instances of rules, such that the leaves are axioms and every assumption of a rule is the conclusion of an immediately preceding rule; the conclusion of the rule at the root is called *provable*.

In the rules defined in the sequel, three dots at the beginning of an r-hypersequent replace, for each rule uniformly, an arbitrary finite multiset of r-sequents, called *side r-sequents*. Furthermore, $\lesseqqgtr$ must be specified as $\leq$ or $<$ for each rule uniformly.

Finally, a *quasiatomic* proposition, or a *quasiatom* for short, is of the form α or $\triangledown\alpha$ for some atom α. An r-sequent or r-hypersequent is called quasiatomic if consisting of quasiatoms only. Moreover, for an atom α, we say that α *is quasi-in* Γ if either α or $\triangledown\alpha$ is in Γ. Finally, we denote by $\Gamma \setminus \alpha$ the multiset resulting from Γ by deleting all occurrences of α and $\triangledown\alpha$ from Γ.

Definition 4.1 The calculus **rHMŁ** consists of a set of *analytic rules*, and a set of *rules for quasiatomic r-hypersequents.* The analytic rules of **rHMŁ** are the following:

$$(\odot \mathrm{l}) \ \frac{\ldots \mid \Gamma, \alpha, \beta \lesseqqgtr \Delta \qquad \ldots \mid \Gamma, \triangledown\alpha \lesseqqgtr \Delta \mid \Gamma, \triangledown\beta \lesseqqgtr \Delta}{\ldots \mid \Gamma, \alpha \odot \beta \lesseqqgtr \Delta}$$

$$(\triangledown\odot \mathrm{l}) \ \frac{\ldots \mid \Gamma, \triangledown\alpha \lesseqqgtr \Delta \mid \Gamma, \triangledown\beta \lesseqqgtr \Delta}{\ldots \mid \Gamma, \triangledown(\alpha \odot \beta) \lesseqqgtr \Delta}$$

$$(\odot \mathrm{r}) \ \frac{\ldots \mid \Gamma \lesseqqgtr \Delta, \alpha, \beta \mid \Gamma \lesseqqgtr \Delta, \triangledown\alpha \qquad \ldots \mid \Gamma \lesseqqgtr \Delta, \alpha, \beta \mid \Gamma \lesseqqgtr \Delta, \triangledown\beta}{\ldots \mid \Gamma \lesseqqgtr \Delta, \alpha \odot \beta}$$

$$(\triangledown\odot \mathrm{r}) \ \frac{\ldots \mid \Gamma \lesseqqgtr \Delta, \triangledown\alpha \qquad \ldots \mid \Gamma \lesseqqgtr \Delta, \triangledown\beta}{\ldots \mid \Gamma \lesseqqgtr \Delta, \triangledown(\alpha \odot \beta)}$$

$$(\rightarrow \mathrm{l}) \ \frac{\ldots \mid \Gamma \lesseqqgtr \Delta \mid \Gamma, \beta \lesseqqgtr \Delta, \alpha \qquad \ldots \mid \Gamma \lesseqqgtr \Delta \mid \beta < \alpha}{\ldots \mid \Gamma, \alpha \rightarrow \beta \lesseqqgtr \Delta}$$

$$(\triangledown\rightarrow \mathrm{l}) \ \frac{\ldots \mid \Gamma \lesseqqgtr \Delta \mid \beta < \alpha \qquad \ldots \mid \Gamma, \triangledown\beta \lesseqqgtr \Delta \mid \alpha \leq \beta}{\ldots \mid \Gamma, \triangledown(\alpha \rightarrow \beta) \lesseqqgtr \Delta}$$

$$(\to\mathrm{r})\quad \frac{\ldots \mid \Gamma \leqq \Delta \qquad \ldots \mid \Gamma, \alpha \leqq \Delta, \beta \mid \alpha \leq \beta}{\ldots \mid \Gamma \leqq \Delta, \alpha \to \beta}$$

$$(\nabla{\to}\mathrm{r})\quad \frac{\ldots \mid \Gamma \leqq \Delta \mid \beta < \alpha \qquad \ldots \mid \Gamma \leqq \Delta, \nabla\beta \mid \alpha \leq \beta}{\ldots \mid \Gamma \leqq \Delta, \nabla(\alpha \to \beta)}$$

$$(\nabla\!\!\nabla\mathrm{l})\quad \frac{\ldots \mid \Gamma, \nabla\alpha \leqq \Delta}{\ldots \mid \Gamma, \nabla\!\!\nabla\alpha \leqq \Delta} \qquad (\nabla\!\!\nabla\mathrm{r})\quad \frac{\ldots \mid \Gamma \leqq \Delta, \nabla\alpha}{\ldots \mid \Gamma \leqq \Delta, \nabla\!\!\nabla\alpha}$$

The rules of **rHMŁ** for quasiatomic r-hypersequents are the following. Here, all assumptions are quasiatomic r-hypersequents, and lower-case Greek letters denote quasiatoms. Moreover, any expression $\nabla\alpha$ in a rule's conclusion, where $\alpha = \nabla\beta$ for some atom β, is understood to be $\nabla\beta$.

$$(\mathrm{A1})\ \varnothing \leq \varnothing \qquad (\mathrm{A2})\ \alpha \leq \alpha \qquad (\mathrm{A3})\ 0 \leq \alpha \qquad (\mathrm{A4})\ 0 < \varnothing$$

$$(\mathrm{EW})\quad \frac{\ldots}{\ldots \mid \Gamma \leqq \Delta} \qquad (\mathrm{EC})\quad \frac{\ldots \mid \Gamma \leqq \Delta \mid \Gamma \leqq \Delta}{\ldots \mid \Gamma \leqq \Delta}$$

$$(\mathrm{Cut}{\leq}/{>})\quad \frac{\ldots \mid \Gamma \leq \Delta \qquad \ldots \mid \Delta < \Gamma}{\ldots},$$ where each of Γ and Δ contains at most one quasiatom

$$(\mathrm{O})\quad \frac{\ldots \mid \Gamma \setminus \alpha \leqq \Delta \setminus \alpha}{\ldots \mid \Gamma \leqq \Delta \mid \nabla\alpha \leq \nabla\beta},$$ where α and β are atoms, and both α and β are quasi-in $\Gamma \cup \Delta$

$$(\nabla\mathrm{l})\quad \frac{\ldots \mid \Gamma, \alpha \leqq \Delta}{\ldots \mid \Gamma, \nabla\alpha \leqq \Delta} \qquad (\nabla\mathrm{lr})\quad \frac{\ldots \mid \beta \leqq \alpha}{\ldots \mid \nabla\beta \leq \nabla\alpha} \qquad (\nabla\mathrm{r})\quad \frac{\ldots \mid \varnothing \leq \alpha}{\ldots \mid \varnothing \leq \nabla\alpha}$$

$$(\mathrm{wl})\quad \frac{\ldots \mid \Gamma \leqq \Delta}{\ldots \mid \Gamma, \alpha \leqq \Delta} \qquad (\mathrm{w0l})\quad \frac{\ldots \mid \Gamma \leq \Delta}{\ldots \mid \Gamma, 0 < \Delta}$$

$$(\mathrm{w}\nabla\mathrm{l})\quad \frac{\ldots \mid \alpha_1, \ldots, \alpha_n \leq \Delta}{\ldots \mid \alpha_1, \ldots, \alpha_n, \nabla\beta < \Delta \mid \nabla\alpha_1 < \nabla\beta \mid \ldots \mid \nabla\alpha_n < \nabla\beta \mid \varnothing \leq \beta},$$
where $n \geq 0$; in case $n = 0$ the r-sequents "$\ldots < \nabla\beta$" are omitted

$$(\mathrm{M})\quad \frac{\ldots \mid \Gamma_1 \leqq \Delta_1 \qquad \ldots \mid \Gamma_2 \leqq \Delta_2}{\ldots \mid \Gamma_1, \Gamma_2 \leqq \Delta_1, \Delta_2} \qquad (\mathrm{S})\quad \frac{\ldots \mid \Gamma_1, \Gamma_2 \leqq \Delta_1, \Delta_2}{\ldots \mid \Gamma_1 \leqq \Delta_1 \mid \Gamma_2 \leqq \Delta_2}$$

$$(\mathrm{S}{<})\quad \frac{\ldots \mid \Gamma, \alpha_1, \ldots, \alpha_n \leq \Delta_1, \Delta_2}{\ldots \mid \Gamma \leq \Delta_1 \mid \alpha_1, \ldots, \alpha_n < \Delta_2 \mid \nabla\alpha_1 < \nabla\beta \mid \ldots \mid \nabla\alpha_n < \nabla\beta},$$
where (i) $n \geq 0$; in case $n = 0$ the r-sequents "$\ldots < \nabla\beta$" are omitted, and (ii) $\beta \in \Gamma \cup \Delta_1$

A proposition α is said to be *provable* in **rHMŁ** if so is the sequent $\varnothing \leq \alpha$.

The choice of this set of rules for **rHMŁ** is inspired by the calculus **rHŁ** for the Łukasiewicz logic, presented in [7]. Indeed, the rules for the introduction of $\rightarrow$ coincide in both calculi, and the rules for $\odot$ are modified according to the fact that the role of the 0 constant is played in the present context by propositions of the form $\triangledown\alpha$. Furthermore, the uniform axioms and structural rules of **rHŁ** are all present in **rHMŁ**. Finally, the two structural rules of **rHŁ** which are not among the uniform rules, but chosen individually for **Ł**, appear in an adapted form in **rHMŁ** as well. Needless to say that, on the other hand, the rules introducing the connective $\triangledown$ are new.

We will define also the notion of validity for r-hypersequents in analogy to the case of Łukasiewicz logic [15]. We will use $(S_k; \odot, \rightarrow, \triangledown, z_k, e)$, $k \geq 1$, the k-fold Łukasiewicz algebra, as well as $(T_k; \cdot, \rightarrow, e)$, the k-fold standard cancellative hoop, and we will consider S_k as a subset of T_k.

Definition 4.2 Let $v \colon \mathcal{P}_{\mathbf{MŁ}} \rightarrow S_k$ be an evaluation of **MŁ**. We say that an r-sequent $\alpha_1, \ldots, \alpha_m \lesseqqgtr \beta_1, \ldots, \beta_n$, where $m, n \geq 0$, is *satisfied* by v if

$$v(\alpha_1) \cdot \ldots \cdot v(\alpha_m) \lesseqqgtr v(\beta_1) \cdot \ldots \cdot v(\beta_n); \tag{4}$$

here, $\cdot$ is the product of T_k, and $\lesseqqgtr$ denotes the order or strict order of T_k, respectively. An r-sequent is called *valid* in **MŁ** if it is satisfied by all evaluations of **MŁ**.

Moreover, we say that an r-hypersequent is *satisfied* by an evaluation v of **MŁ** if at least one of its elements is satisfied by v. An r-hypersequent is *valid* in **MŁ** if it is satisfied by all evaluations of **MŁ**.

Clearly, the product of an empty set of elements of some T_k is assumed to be $e = (0, 0)$.

In this section, the validity of r-sequents and r-hypersequents will always mean validity in **MŁ**.

In the remainder of the section, we will prove the soundness and completeness of **rHMŁ**. To see that **rHMŁ** is sound for **MŁ** is tedious, but not difficult. In contrast, the proof that **rHMŁ** is actually complete, is quite involved. This proof is split into a series of lemmas, whose contents may roughly be summarize as follows: (1) The analytic rules are invertible. (2) The backwards application of the analytic rules ends after finitely many steps with the presence of quasiatomic r-hypersequents. (3) There are two admissible invertible rules for quasiatomic r-hypersequents such that, applying them finitely many times backwards, we are led to quasiatomic r-hypersequents belonging to one of two types: those whose validity is derivable from a statement on bounded linear orders; and those whose validity is derivable by restricting to evaluations in the Łukasiewicz algebra. (4) r-hypersequents of the former type are provable in **rHMŁ**. (5) r-hypersequents of the latter type are provable in **rHMŁ**.

In what follows, we say that a rule *preserves validity* if whenever for some evaluation v, all assumptions are satisfied by v, also the conclusion is satisfied by v. We say that a rule is *invertible* if for any evaluation v, all assumptions are satisfied by v if and only if the conclusion is satisfied by v.

Our first lemma contains the statement that **rHMŁ** is sound.

Lemma 4.3 *All rules of* **rHMŁ** *preserve validity. Moreover, all analytic rules of* **rHMŁ** *as well as the rules* (Cut$\leq$/$>$) *and* (∇r) *are invertible.*

Proof. Let v be any evaluation of the propositions of **MŁ** in some S_k, $k \geq 1$. In case of the rules (EW) and (Cut$\leq$/$>$), if the assumptions are satisfied by v, one of the side r-sequents and consequently the conclusion is satisfied by v. In case of (Cut$\leq$/$>$), the converse evidently holds as well. So (EW) preserves validity, and (Cut$\leq$/$>$) is invertible.

For the remaining rules, we may w.l.o.g. assume that there are no side r-sequents.

By Lemma 3.5(i), the two assumptions of ($\odot$l) are satisfied by v iff the conclusion is satisfied by v. In particular, ($\odot$l) is invertible. By Lemma 3.5(ii), the same applies to ($\to$r). Taking the equivalences stated in Lemma 3.5(i),(ii) in their negated form, we see that also ($\odot$r) and ($\to$l) are invertible.

From (2) and the monotonicity of ∇, we see that the assumption of ($\nabla\odot$l) is satisfied by v iff its conclusion is satisfied by v. So ($\nabla\odot$l), and consequently also ($\nabla\odot$r), are invertible rules. Moreover, it follows from (3) that ($\nabla\to$l), and consequently also ($\nabla\to$r), is invertible. The invertibility of ($\nabla\!\!\nabla$l) and ($\nabla\!\!\nabla$r) is evident.

We next consider the rules manipulating quasiatomic r-hypersequents. That the axioms (A1)–(A4) are satisfied by v, is clear; in case of (A4), recall that an empty r-sequent is mapped to e. For the rule (EC), the assertion is clear as well.

The fact that the conclusion of (O) is satisfied by v if the assumption is satisfied by v, follows from Lemma 3.6. The assertion is moreover clear for (∇l), (∇lr), (wl), and (w0l). Furthermore, since, for any $a \in S_k$, we have $a = e$ iff $\nabla a = e$, (∇r) is invertible. For (w∇l), the case $n = 0$ is evident, and if $n \geq 1$, we argue similarly as for (w0l). For (M) and (S), we make use of Lemma 3.7(i),(ii). Finally, the case $n = 0$ of (S$<$) is trivial; for the case $n \geq 1$, we make use of Lemma 3.7(iii). $\square$

Lemma 4.4 *Applying the analytic rules of* **rHMŁ** *successively upwards to some r-hypersequent, terminates with quasiatomic r-hypersequents.*

Proof. We shall slightly modify the proof of [7, Proposition 1]. Namely, define the complexity of propositions as follows: $c_P(\alpha) = 1$ for an atom α; and $c_P(\alpha \odot \beta) = c_P(\alpha \to \beta) = c_P(\alpha) + c_P(\beta) + 1$, $c_P(\nabla\alpha) = c_P(\alpha)$ for arbitrary propositions α, β. For an r-sequent $\mathcal{S}$, let $c_S(\mathcal{S}) = \{c_P(\alpha) \colon \alpha$ is contained in the antecedent or succedent of $\mathcal{S}\}$, understood as a multiset. For an r-hypersequent $\mathcal{H}$, let $c_H(\mathcal{H}) = \{c_S(\mathcal{S}) : \ \mathcal{S}$ is contained in $\mathcal{H}\}$, again understood as a multiset.

For finite multisets of natural numbers M and N, we define $M <_S N$ if $M = (N \setminus N') \cup N''$, where $N' \subseteq N$ is non-empty and for each $m \in N''$ there is an $n \in N'$ such that $m < n$. For finite multisets of finite multisets of natural numbers $\bar{M}$ and $\bar{N}$, we analogously define $\bar{M} <_H \bar{N}$, using the order $<_S$.

Then for any of the assumptions $\mathcal{H}_1$ and the conclusion $\mathcal{H}_2$ of an analytic rule, we have $c_H(\mathcal{H}_1) <_H c_H(\mathcal{H}_2)$ if the rule is not ($\nabla\!\!\nabla$l) or ($\nabla\!\!\nabla$r), else $c_H(\mathcal{H}_1) = c_H(\mathcal{H}_2)$. Moreover, by an argument based on König's lemma, we see that there are no infinite strictly decreasing $<_H$-chains. Because ($\nabla\!\!\nabla$l) and ($\nabla\!\!\nabla$r) can be backwards applied to an r-hypersequent only finitely many times, the assertion follows. $\square$

The remainder of the **rHMŁ**-completeness proof concerns quasiatomic r-hypersequents. In this part, the connective ∇, which actually was introduced to make invertible rules for the introduction of the conjunction possible, will again play the crucial role. By means of the cut rule $(\mathrm{Cut}{\leq}/{>})$, we will use ∇ to distinguish the cases that two variables have values in the same Archimedean class or in different ones.

Lemma 4.5 *The following rule is admissible in* **rHMŁ***:*

$(\mathrm{ExtCut}{<}/{>}/{=})$

$$\frac{\ldots \mid \Gamma\backslash\alpha \leqq \Delta\backslash\alpha \mid \nabla\alpha \leq \nabla\beta \qquad \ldots \mid \Gamma\backslash\beta \leqq \Delta\backslash\beta \mid \nabla\beta \leq \nabla\alpha \qquad \ldots \mid \Gamma \leqq \Delta \mid \nabla\alpha < \nabla\beta \mid \nabla\beta < \nabla\alpha}{\ldots \mid \Gamma \leqq \Delta},$$

where α and β are atoms quasi-in $\Gamma \cup \Delta$;

here, the assumptions are assumed to be quasiatomic.

Moreover, $(\mathrm{ExtCut}{<}/{>}/{=})$ *is invertible.*

Before beginning the proof, we insert one remark. The rule $(\mathrm{ExtCut}{<}/{>}/{=})$ is, according to Lemma 4.5, invertible; however, $(\mathrm{ExtCut}{<}/{>}/{=})$ is composed of several rules of **rHMŁ** which themselves are not necessarily invertible.

Proof. To the conclusion $\ldots \mid \Gamma \leqq \Delta$, apply backwards $(\mathrm{Cut}{\leq}/{>})$ twice and then (EW); this gives a rule $(\mathrm{ExtCut}{<}/{>}/{=}')$ arising from $(\mathrm{ExtCut}{<}/{>}/{=})$ by omitting "$\setminus \alpha$" and "$\setminus \beta$". By distinguishing equality or inequality of $\nabla\alpha$ and $\nabla\beta$ under some evaluation, we easily see that $(\mathrm{ExtCut}{<}/{>}/{=}')$ is invertible.

Furthermore, the rule

$$(\mathrm{O'}) \quad \frac{\ldots \mid \Gamma \setminus \alpha \leqq \Delta \setminus \alpha \;\mid\; \nabla\alpha \leq \nabla\beta}{\ldots \mid \Gamma \leqq \Delta \;\mid\; \nabla\alpha \leq \nabla\beta},$$

where α and β are atoms quasi-in $\Gamma \cup \Delta$, is admissible by (O) and (EC). Moreover, we conclude from Lemma 3.6 that (O') is invertible. It follows that $(\mathrm{ExtCut}{<}/{>}/{=})$ is admissible and invertible. □

Lemma 4.6 *The following rules are admissible in* **rHMŁ***:*

$$(\mathrm{ExtCut}{=}e/{<}e) \quad \frac{\ldots \mid \Gamma \leqq \Delta \;\mid\; \alpha < \emptyset \qquad \ldots \mid \Gamma \leqq \Delta \;\mid\; \emptyset \leq \alpha}{\ldots \mid \Gamma \leqq \Delta}, \quad \text{where } \alpha \text{ is an atom;}$$

$$(\nabla\mathrm{l}{<}\emptyset) \quad \frac{\ldots \mid \alpha < \emptyset}{\ldots \mid \nabla\alpha < \emptyset}; \qquad (\emptyset{<}) \quad \frac{\ldots}{\ldots \mid \emptyset < \Gamma}; \qquad (\nabla\mathrm{r}) \quad \frac{\ldots \mid \emptyset \leq \alpha}{\ldots \mid \emptyset \leq \nabla\alpha};$$

here, all assumptions are assumed to be quasiatomic.

Moreover, the rules $(\mathrm{ExtCut}{=}e/{<}e)$*,* $(\nabla\mathrm{l}{<}\emptyset)$*,* $(\emptyset{<})$*, and* $(\nabla\mathrm{r})$ *are invertible.*

Furthermore, any quasiatomic r-hypersequent of the form $\ldots \mid \Gamma \leq \emptyset$ *is provable in* **rHMŁ***.*

Proof. $(\mathrm{ExtCut}{=}e/{<}e)$ is an instance of $(\mathrm{Cut}{\leq}/{>})$, and invertible by Lemma 4.3.

The rule $(\nabla\mathrm{l}{<}\emptyset)$ is an instance of $(\nabla\mathrm{l})$, and obviously invertible. $(\emptyset{<})$ is admissible by (EW), and since an r-sequent of the form $\emptyset < \Gamma$ is not satisfied by any evaluation, $(\emptyset{<})$ is invertible. The rule $(\nabla\mathrm{r})$ is contained in **rHMŁ**, and invertible by Lemma 4.3.

Finally, a quasiatomic r-hypersequent $\ldots \mid \Gamma \leq \emptyset$ is proved from (A1) by means of (wl) and (EW). $\square$

We will introduce some auxiliary notions to simplify the understanding of the subsequent steps.

First of all, call an r-hypersequent *trivial* if it contains an r-sequent with an empty succedent. Note that any trivial r-hypersequent is provable by Lemma 4.6.

Furthermore, let us call an r-sequent *basic* if it is of the form

$$\nabla\alpha \leqq \nabla\beta \quad \text{or} \quad \emptyset \leq \alpha \quad \text{or} \quad \alpha < \emptyset,$$

where α and β are atoms. Moreover, a quasiatomic r-hypersequent will be called *basic* if consisting of basic r-sequents only. Finally, let $\mathcal{H}$ be any quasiatomic r-hypersequent; we will denote by $\mathcal{H}_b$ the r-hypersequent arising from $\mathcal{H}$ by deleting all non-basic r-sequents.

Furthermore, we will call a quasiatomic r-hypersequent $\mathcal{H}$ *with specified Archimedean classes* if the following holds: (i) If a non-basic r-sequent contains the two distinct atoms α and β, then $\mathcal{H}$ contains the r-sequents $\nabla\alpha < \nabla\beta$ and $\nabla\beta < \nabla\alpha$. (ii) For each variable α appearing in a non-basic r-sequent of $\mathcal{H}$, $\mathcal{H}$ contains either the r-sequent $\emptyset \leq \alpha$ or $\alpha < \emptyset$. (iii) $\mathcal{H}$ is non-trivial and does not contain any r-sequent of the form $\emptyset < \Gamma$.

Lemma 4.7 *To some non-trivial quasiatomic r-hypersequent, we can apply the rules* $(\mathrm{ExtCut}{<}/{>}/{=})$, $(\mathrm{ExtCut}{=}\, e/{<}\, e)$, $(\nabla\mathrm{l}{<}\, \emptyset)$, $(\emptyset <)$, *and* $(\nabla\mathrm{r})$ *successively upwards, to terminate with quasiatomic r-hypersequents with specified Archimedean classes.*

Proof. Let $\mathcal{H}$ be a non-trivial quasiatomic r-hypersequent. We may remove all r-sequents of the form $\emptyset < \Gamma$ by backwards applications of $(\emptyset{<})$. So we can safely assume that $\mathcal{H}$ fulfills condition (iii) for an r-hypersequent to be with specified Archimedean classes.

Write then $\mathcal{H} = \Gamma_1 \leqq \Delta_1 | \ldots | \Gamma_n \leqq \Delta_n | \mathcal{H}_b$, where $n \geq 0$. If $n = 0$, then $\mathcal{H}$ is already an r-hypersequent with specified Archimedean classes; so let $n \geq 1$.

Let now (i, α, β) a triple consisting of a number $i \in \{1, \ldots, n\}$ and two (not necessarily distinct) variables α and β such that α and β are contained in $\Gamma_i \cup \Delta_i$. Call then (i, α, β) an unmet requirement of $\mathcal{H}$ if either $\alpha = \beta$ and $\mathcal{H}$ contains neither $\emptyset \leq \alpha$ nor $\alpha < \emptyset$, or $\alpha \neq \beta$ and $\mathcal{H}$ does not contain $\nabla\alpha < \nabla\beta$ or $\nabla\beta < \nabla\alpha$.

Let there be k unmet requirements in $\mathcal{H}$. Now, for an arbitrary unmet requirement (i, α, β), we may apply $(\mathrm{ExtCut}{<}/{>}/{=})$ or $(\nabla\mathrm{l}{<}\emptyset)$ or $(\nabla\mathrm{r})$ or $(\mathrm{ExtCut}{=}e/{<}e)$ backwards with the following effect: In each of the assumptions, by keeping the numbering of the non-basic r-sequents, the triple (i, α, β) is no longer an unmet requirement; and there are no additional unmet requirements, compared to $\mathcal{H}$.

It follows that continuing to do similar steps with respect to unmet requirements of r-hypersequents at the leaves, the length of the resulting tree is at most k. In particular, the process terminates with r-hypersequents with specified Archimedean classes. $\square$

Our next task is to construct a proof of a valid non-trivial quasiatomic r-hypersequent $\mathcal{H}$ with specified Archimedean classes. We distinguish two cases: Either $\mathcal{H}_b$ is valid, or $\mathcal{H}_b$ is not valid. In the next two lemmas, we deal with the first case.

Lemma 4.8 *The following rules are admissible in* **rHMŁ***:*

$$(\triangledown\text{l-})\ \frac{\ldots \mid \triangledown\alpha < \emptyset}{\ldots \mid \alpha < \emptyset}; \quad (\triangledown\text{r-})\ \frac{\ldots \mid \emptyset \leq \triangledown\alpha}{\ldots \mid \emptyset \leq \alpha}.$$

here, the assumption is quasiatomic and α *is an atom.*

Proof. We show first that $\emptyset \leq \alpha \mid \alpha < \emptyset$ is provable. Indeed, (S<) applied to $\alpha \leq \alpha$ results in $\emptyset \leq \alpha \mid \alpha < \emptyset \mid \triangledown\alpha < \triangledown\alpha$. Furthermore, $\triangledown\alpha \leq \triangledown\alpha$ is an instance of (A2); so by an applications of (EW) and (Cut≤/>), we derive $\emptyset \leq \alpha \mid \alpha < \emptyset$.

To see that ($\triangledown$l-) is admissible, note that from $\ldots \mid \triangledown\alpha < \emptyset$, we derive $\ldots \mid \alpha < \emptyset \mid \triangledown\alpha < \emptyset$ by (EW), and $\emptyset \leq \alpha \mid \alpha < \emptyset$ gives $\emptyset \leq \triangledown\alpha \mid \alpha < \emptyset$ by ($\triangledown$r). So applying (Cut≤/>) to $\ldots \mid \alpha < \emptyset \mid \triangledown\alpha < \emptyset$ and $\ldots \mid \alpha < \emptyset \mid \emptyset \leq \triangledown\alpha$, we derive $\ldots \mid \alpha < \emptyset$ as desired.

For the second rule ($\triangledown$r-), derive from $\ldots \mid \emptyset \leq \triangledown\alpha$ by (EW) $\ldots \mid \emptyset \leq \alpha \mid \emptyset \leq \triangledown\alpha$. Furthermore, from $\emptyset \leq \alpha \mid \alpha < \emptyset$, derive $\emptyset \leq \alpha \mid \triangledown\alpha < \emptyset$ by ($\triangledown$l). So an application of (Cut≤/>) gives $\ldots \mid \emptyset \leq \alpha$. □

Lemma 4.9 *Let* $\mathcal{H}$ *be a basic r-hypersequent. If* $\mathcal{H}$ *is valid, then* $\mathcal{H}$ *is provable in* **rHMŁ***.*

Proof. Let $\mathcal{H}$ be valid, and let $\mathcal{H}'$ be the r-hypersequent arising from $\mathcal{H}$ by replacing all r-sequents of the form $\emptyset \leq \alpha$ by $\emptyset \leq \triangledown\alpha$, and all r-sequents of the form $\alpha < \emptyset$ by $\triangledown\alpha < \emptyset$. Obviously, $\mathcal{H}'$ is valid as well. We shall prove that $\mathcal{H}'$ is provable in **rHMŁ**; the provability of $\mathcal{H}$ will then follow by Lemma 4.8.

The validity of $\mathcal{H}'$ may be understood in the obvious way as a statement on bounded linear orders, where $\triangledown 0$ is interpreted as the bottom element and the empty multiset is interpreted as the top element. So by Lemma 3.8, $\mathcal{H}'$ is derivable by external weakening from the following type of r-hypersequents:

$$\triangledown\alpha_1 \lesseqqgtr \triangledown\alpha_2 \mid \triangledown\alpha_2 \lesseqqgtr \triangledown\alpha_3 \mid \ldots \mid \triangledown\alpha_{n-1} \lesseqqgtr \triangledown\alpha_n \mid \triangledown\alpha_n \leq \triangledown\alpha_1, \tag{5}$$

where each $\lesseqqgtr$ is chosen independently as $\leq$ or $<$;

$$\triangledown 0 < \triangledown\alpha_1 \mid \triangledown\alpha_1 < \triangledown\alpha_2 \mid \ldots \mid \triangledown\alpha_{n-2} < \triangledown\alpha_{n-1} \mid \triangledown\alpha_{n-1} \leq \triangledown\alpha_n; \tag{6}$$

$$\triangledown\alpha_1 \leq \triangledown\alpha_2 \mid \triangledown\alpha_2 < \triangledown\alpha_3 \mid \ldots \mid \triangledown\alpha_{n-1} < \triangledown\alpha_n \mid \triangledown\alpha_n < \emptyset; \tag{7}$$

$$\emptyset \leq \triangledown\alpha_2 \mid \triangledown\alpha_2 < \triangledown\alpha_3 \mid \ldots \mid \triangledown\alpha_{n-1} < \triangledown\alpha_n \mid \triangledown\alpha_n < \emptyset; \tag{8}$$

$$\triangledown 0 < \triangledown\alpha_2 \mid \triangledown\alpha_2 < \triangledown\alpha_3 \mid \ldots \mid \triangledown\alpha_{n-1} < \triangledown\alpha_n \mid \triangledown\alpha_n < \emptyset; \tag{9}$$

the case $n = 1$ means $\triangledown\alpha_1 \leq \triangledown\alpha_1$, $\triangledown 0 \leq \triangledown\alpha_1$, $\triangledown\alpha_1 \leq \emptyset$, $\emptyset \leq \emptyset$, $\triangledown 0 < \emptyset$, respectively.

Let us see how these r-hypersequents are proved in **rHMŁ**. The cases $n = 1$ are immediate from (A1)–(A4). Let $n \geq 2$. By (S<) and (EC), we have

$$\frac{\triangledown\alpha_1, \ldots, \triangledown\alpha_n \leq \triangledown\alpha_2, \ldots, \triangledown\alpha_n, \triangledown\alpha_1}{\triangledown\alpha_1 < \triangledown\alpha_2 \mid \triangledown\alpha_2, \ldots, \triangledown\alpha_n \leq \triangledown\alpha_3, \ldots, \triangledown\alpha_n, \triangledown\alpha_1},$$

and if we apply (S) instead, the "<" is replaced by "≤". Continuing this way, we get (5).

When replacing in this proof the left occurrences of $\triangledown\alpha_1$ by $\triangledown 0$, we derive – up to the numbering of the variables – (6).

(7) is evidently derivable from (8) by (wl). For (8), we proceed as follows. From (A2), we have $\emptyset \leq \triangledown\alpha_n \mid \triangledown\alpha_n < \emptyset$ by (S<); note that we may drop $\triangledown\alpha_n < \triangledown\alpha_n$ by (Cut≤/>). Similarly, we derive $\emptyset \leq \triangledown\alpha_{n-1} \mid \triangledown\alpha_{n-1} < \triangledown\alpha_n \mid \triangledown\alpha_n < \emptyset$, and so forth.

(9) follows by (w0l), ($\triangledown$l), and repeatedly (S). □

As a corollary, we insert the following fact.

Lemma 4.10 *The following rule is admissible in* **rHMŁ***:*

$$\text{(Red)}\ \frac{\triangledown\gamma_1 < \triangledown\gamma_2 \ \mid\ \triangledown\gamma_2 < \triangledown\gamma_3 \ \mid\ \ldots\ \mid\ \triangledown\gamma_{k-1} < \triangledown\gamma_k \ \mid\ \triangledown\gamma_1 < \triangledown\gamma_k}{\triangledown\gamma_1 < \triangledown\gamma_2 \ \mid\ \triangledown\gamma_2 < \triangledown\gamma_3 \ \mid\ \ldots\ \mid\ \triangledown\gamma_{k-1} < \triangledown\gamma_k}$$

where $\gamma_1, \ldots, \gamma_k$, $k \geq 3$, are atoms.

Proof. The basic r-hypersequent $\triangledown\gamma_1 < \triangledown\gamma_2 \mid \triangledown\gamma_2 < \triangledown\gamma_3 \mid \ldots \mid \triangledown\gamma_{k-1} < \triangledown\gamma_k \mid \triangledown\gamma_k \leq \triangledown\gamma_1$ is valid and consequently derivable by Lemma 4.9. So (Red) is admissible, based on an application of (Cut≤/>). □

Again, let $\mathcal{H}$ be a non-trivial quasiatomic r-hypersequent with specified Archimedean classes. We now turn to the case that $\mathcal{H}_b$, the sub-r-hypersequent consisting of the basic r-sequents alone, is not valid. We shall see that in this case, $\mathcal{H}$ is composed of r-hypersequents of the following type.

Namely, let us call an r-hypersequent $\mathcal{H}$ *of type Łukasiewicz* if the following holds: (i) For any pair α, β of distinct atoms appearing in $\mathcal{H}$, there are atoms $\gamma_1, ..., \gamma_l$ such that $\mathcal{H}$ contains the r-sequents $\triangledown\alpha < \triangledown\gamma_1$, $\triangledown\gamma_1 < \triangledown\alpha$, $\triangledown\gamma_1 < \triangledown\gamma_2$, $\triangledown\gamma_2 < \triangledown\gamma_1$, …, $\triangledown\beta < \triangledown\gamma_l$, $\triangledown\gamma_l < \triangledown\beta$. (ii) For each variable α appearing in a non-basic r-sequent of $\mathcal{H}$, $\mathcal{H}$ contains either the r-sequent $\emptyset \leq \alpha$ or $\alpha < \emptyset$. (iii) $\mathcal{H}$ is non-trivial and does not contain any r-sequent of the form $\emptyset < \Gamma$.

Lemma 4.11 *Let $\mathcal{H}$ be a quasiatomic r-hypersequent with specified Archimedean classes. Assume that $\mathcal{H}$ is valid, but $\mathcal{H}_b$ is not valid. Then $\mathcal{H} = \mathcal{K}_1|...|\mathcal{K}_n|\mathcal{S}$, where* (i) *$\mathcal{K}_i$ is, for every i, of type Łukasiewicz,* (ii) *for each distinct indices i, j, $\mathcal{K}_i$ and $\mathcal{K}_j$ have no atom in common,* (iii) *$\mathcal{S}$ consists of r-sequents of the form $\triangledown\alpha \lesseqgtr \triangledown\beta$, where the atoms α and β do not both appear in any $\mathcal{K}_i$.*

Proof. For two atoms α and β, put $\alpha \approx' \beta$ if $\alpha = \beta$ or else the r-sequents $\triangledown\alpha < \triangledown\beta$ and $\triangledown\beta < \triangledown\alpha$ are contained in $\mathcal{H}$, and let $\approx$ be the transitive closure of $\approx'$. Let $V_1, \ldots, V_n$ be the $\approx$-equivalence classes of V. Furthermore, let, for each $i = 1, ..., n$, $\mathcal{K}_i$ consist of exactly those r-sequents in $\mathcal{H}$ all of whose variables are contained in V_i.

$V_1, \ldots, V_n$ are a partition of the set of atoms appearing in $\mathcal{H}$; hence the $\mathcal{K}_i$ are pairwise without common atoms and, in particular, disjoint. It is furthermore clear that each $\mathcal{K}_i$ is of type Łukasiewicz. So (i) and (ii) hold.

Choose now $\mathcal{S}$ such that $\mathcal{H} = \mathcal{K}_1|...|\mathcal{K}_n|\mathcal{S}$. Then $\mathcal{S}$ is basic; indeed, for any non-basic r-sequent in $\mathcal{H}$, all its variables are contained in one of the sets V_i because $\mathcal{H}$ is with specified Archimedean classes. Taking into account that each r-sequent in $\mathcal{S}$ must contain at least two atoms, (iii) follows. $\Box$

Lemma 4.12 *Let $\mathcal{H}$ be a quasiatomic r-hypersequent with specified Archimedean classes. Assume that $\mathcal{H}$ is valid, but $\mathcal{H}_b$ is not valid. Then there is a valid quasiatomic r-hypersequent of type Łukasiewicz contained in $\mathcal{H}$.*

Proof. Let $\mathcal{H} = \mathcal{K}_1|...|\mathcal{K}_n|\mathcal{S}$ according to Lemma 4.11.

Furthermore, let v be an evaluation such that $\mathcal{H}_b$ is not satisfied by v. Then, for each i, the atoms contained in $\mathcal{K}_i$ are mapped all to the same Archimedean class. Furthermore, for any further evaluation v' such that $v(\alpha) \sim v'(\alpha)$ for every atom α appearing in $\mathcal{H}$, $\mathcal{H}_b$ is not satisfied by v' either.

Assume now that for no i, $\mathcal{K}_i$ is valid. For each i, let v_i be an evaluation of the variables in $\mathcal{K}_i$ such that $\mathcal{K}_i$ is not satisfied by v_i. Note that v_i maps all atoms contained in $\mathcal{K}_i$ to the same Archimedean class. So, w.l.o.g. we may assume that the range of v_i coincides with the range of v and that, for any atom α appearing in $\mathcal{K}_i$, $v_i(\alpha) \sim v(\alpha)$.

Define then the evaluation v' such that $v'(\alpha) = v_i(\alpha)$ for each variable α contained in $\mathcal{K}_i$, and $v'(\alpha) = v(\alpha)$ for each of the remaining variables, i.e. those contained only in $\mathcal{S}$. Then $\mathcal{H}$ is not satisfied by v', in contradiction to the assumption. $\Box$

Lemma 4.13 *Let $\mathcal{H}$ be a quasiatomic r-hypersequent of type Łukasiewicz. If $\mathcal{H}$ is valid, then $\mathcal{H}$ is provable in* **rHMŁ**.

Proof. Let $\mathcal{H}$ be valid, and let $\alpha_1, \ldots, \alpha_n$ be the atoms appearing in $\mathcal{H}$. If $\mathcal{H}_b$ is valid, $\mathcal{H}$ is provable in **rHMŁ** by Lemma 4.9.

Let $\mathcal{H}_b$ be not valid. Since $\emptyset \leq \emptyset$ or $\emptyset < \emptyset$ is not in $\mathcal{H}$, at least one atom appears in $\mathcal{H}$ which is contained in a non-basic r-sequent.

For any atom α_i appearing in a non-basic r-sequent, either $\alpha_i < \emptyset$ or $\emptyset \leq \alpha_i$ is in $\mathcal{H}$. Moreover, for any (not necessarily distinct) atoms appearing in $\mathcal{H}$, e.g. α_1 and α_2, the r-sequents $\alpha_1 < \emptyset$ and $\emptyset \leq \alpha_2$ cannot be both in $\mathcal{H}$, because by assumption r-sequents $\triangledown\alpha_1 < \triangledown\gamma_1$, $\triangledown\gamma_1 < \triangledown\alpha_1$, $\triangledown\gamma_1 < \triangledown\gamma_2$, $\triangledown\gamma_2 < \triangledown\gamma_1, \ldots,$ $\triangledown\alpha_2 < \triangledown\gamma_k$, $\triangledown\gamma_k < \triangledown\alpha_2$ are contained in $\mathcal{H}$, whence $\mathcal{H}_b$ would be valid.

Assume that an r-sequent $\alpha_i < \emptyset$ is contained in $\mathcal{H}$. Then 0 is not among the atoms appearing in $\mathcal{H}$; indeed, otherwise an r-hypersequent $\triangledown 0 < \triangledown\gamma_1 \mid \triangledown\gamma_1 < \triangledown\gamma_2 \mid ... \mid \triangledown\gamma_{k-1} < \triangledown\alpha_i \mid \alpha_i < \emptyset$, $k \geq 0$, would be in $\mathcal{H}$, so that $\mathcal{H}_b$ would be valid.

So all atoms are variables. Furthermore, $\mathcal{H}$ must contain a non-basic r-sequent whose relational symbol is $\leq$; indeed, the basic r-sequents all have the relational symbol $<$, and so $\mathcal{H}$ would not be satisfied by the evaluation mapping each variable to e. Let $\Gamma \leq \alpha_{i_1}, ..., \alpha_{i_k}$ be this r-sequent of $\mathcal{H}$; note that $k \geq 1$.

In this case, we prove $\mathcal{H}$ as follows. Apply (S<) k-times to $\alpha_{i_1}, ..., \alpha_{i_k} \leq \alpha_{i_1}, ..., \alpha_{i_k}$, to derive $\alpha_{i_1} < \emptyset \mid ... \mid \alpha_{i_k} < \emptyset \mid \emptyset \leq \alpha_{i_1}, ..., \alpha_{i_k}$. Furthermore, to the last r-sequent, apply (wl) w.r.t. the atoms contained in Γ. Applying finally (EW) if necessary, we get $\mathcal{H}$.

Let us assume from now on that no r-sequent of the form $\alpha_i < \emptyset$ is contained in $\mathcal{H}$, but, for all atoms α_i in the non-basic r-sequents, the r-sequent $\emptyset \leq \alpha_i$ is in $\mathcal{H}$. Let $\mathcal{H}'$ be the r-hypersequent arising from $\mathcal{H}$ by deleting all r-sequents $\triangledown\alpha_i < \triangledown\alpha_j$ for any i, j, and by deleting all r-sequents $\emptyset \leq \alpha_j$ such that α_j does not appear in a non-basic r-sequent of $\mathcal{H}$. Furthermore, let $\mathcal{H}''$ be the r-hypersequent arising from $\mathcal{H}'$ by replacing all occurrences of $\triangledown\alpha_i$ for any i by 0. Then $\mathcal{H}''$ is a valid r-hypersequent of the Łukasiewicz logic **Ł**, that is, $\mathcal{H}''$ is satisfied by all evaluations into S_1.

Consequently, there is a derivation D of $\mathcal{H}''$ in the calculus **rHŁ** of [7]; D uses the rules (A1)–(A4), (EW), (EC), (wl), (w0l), (M), (S) where $\lesseqqgtr$ is chosen $\leq$, and the rule (S') $\frac{\ldots \mid \Gamma_1,\Gamma_2 \leq \Delta_1,\Delta_2}{\ldots \mid \Gamma_1 \leq \Delta_1 \mid \Gamma_2 < \Delta_2}$.

On the set of propositions appearing in the proof D, we next establish a binary relation expressing that an occurrence of one proposition is the *descendant* of another one or, conversely, the latter is the *ancestor* of the former. We do so in a way explained e.g. in [6, Section 1.2.3]; we do not repeat the details here.

In $\mathcal{H}''$, we consider successively every constant 0 which is the result of a change in $\mathcal{H}'$. Replace this constant 0 together with all its ancestors in D by the original quasiatom, that is, by one of $\triangledown\alpha_1$, ..., $\triangledown\alpha_n$. As a result, we will recover $\mathcal{H}'$ as the last r-hypersequent, but the tree above it will in general not be a proof in **rHMŁ**.

Consider next successively all rules in which (at least) one of $\triangledown\alpha_1, \ldots, \triangledown\alpha_n$ is introduced. If this rule is not (EW) or (wl), it is:

(A2), which originally, i.e. in D, was $0 \leq 0$.

Assume first that this r-sequent is now of the form $\triangledown\alpha_i \leq 0$, where α_i is an variable. Then add the additional r-sequent $\triangledown 0 < \triangledown\alpha_i$, to get $\triangledown\alpha_i \leq 0 \mid \triangledown 0 < \triangledown\alpha_i$. Above this r-hypersequent, we add a proof based on (S<). Furthermore, we add $\triangledown 0 < \triangledown\alpha_i$ to all subsequent r-hypersequents as well as to each further assumption used in a subsequent rule (M). In each of the latter cases, we add an application of (EW).

If this r-sequent is $\triangledown 0 \leq 0$, add a proof by (A2) and ($\triangledown$l).

Assume next that the r-sequent is $0 \leq \triangledown\alpha_i$. This is an instance of (A3).

Assume that we are given $\triangledown\alpha_i \leq \triangledown\alpha_i$. This is still an instance of (A2).

Finally, assume that we are given $\triangledown\alpha_i \leq \triangledown\alpha_j$, where $i \neq j$. Then we add $\triangledown\alpha_j < \triangledown\alpha_i$ and proceed according to the explanations for the first case.

(A3), which in D was $0 \leq \alpha_j$. Assume it is now of the form $\triangledown\alpha_i \leq \alpha_j$. If $i = j$, we add a proof based on ($\triangledown$l). If $i \neq j$, we add $\triangledown\alpha_j < \triangledown\alpha_i$ to this leaf as well as to the subsequent steps according to the explanations for (A2). Above $\triangledown\alpha_i \leq \alpha_j \mid \triangledown\alpha_j < \triangledown\alpha_i$, we add a proof using (S<).

(A4), which in D was $0 < \emptyset$. It is now of the form $\triangledown\alpha_i < \emptyset$. Let α_i be a variable. Then add $\emptyset \leq \alpha_i$ to this leaf as well as to the subsequent steps according to the explanations for (A2). We add a proof of $\triangledown\alpha_i < \emptyset \mid \emptyset \leq \alpha_i$ by (S<).

If, otherwise, $\alpha_i = 0$, add a proof from (A4) by ($\triangledown$l).

(w0l). If $\triangledown 0$ is introduced here, replace it by 0 and add an application of ($\triangledown$l). Otherwise, we apply the rule (w$\triangledown$l) instead. Among the r-sequents appearing additionally, we delete those of the form $\triangledown\alpha_i < \triangledown\alpha_i$ by (Cut$\leq$/$>$); the others are added to all subsequent steps according to the explanation for (A2).

Next, consider the usage of the rule (S'). The case $\Gamma_1 \cup \Delta_1 = \emptyset$ can safely be excluded. So we replace (S') by (S$<$), and we treat additionally appearing r-sequents to the subsequent steps according to the explanation for the rule (w0l).

The result will be a proof of $\mathcal{H}'$ in **rHMŁ**. We continue this proof as follows. By means of (EW), we add all r-sequents $\triangledown\alpha_i < \triangledown\alpha_j$ and $\emptyset \leq \alpha_j$ previously deleted from $\mathcal{H}$, in case they are not already present. Applying (EC) and (EW) if necessary, we derive in this way an r-hypersequent $\mathcal{H}'''$ with the following properties: $\mathcal{H}$ is contained in $\mathcal{H}'''$; and any r-sequent contained in $\mathcal{H}'''$, but not in $\mathcal{H}$, is of the form $\triangledown\alpha_i < \triangledown\alpha_j$ such that there are r-sequents $\triangledown\alpha_i < \triangledown\gamma_1,\ \triangledown\gamma_1 < \triangledown\gamma_2,\ \ldots,\ \triangledown\gamma_l < \triangledown\alpha_j$ in $\mathcal{H}$. So by means of the rule (Red) of Lemma 4.10, we may derive $\mathcal{H}$ from $\mathcal{H}'''$. □

The proof of our main assertion is complete.

Theorem 4.14 *The calculus* **rHMŁ** *is sound and complete for* **MŁ**: *A proposition α is valid in* **MŁ** *if and only if α is provable in* **rHMŁ**.

As an immediate corollary, we have: A proposition α in the restricted language $\odot$, $\rightarrow$, and 0, is valid in the Basic Logic **BL** if and only if α is provable in **rHMŁ**.

Taking into account the results of [7] on the logic **rHŁ**, we can actually say more than what is stated in Theorem 4.14: For each proposition φ of **MŁ**, we can, by means of the calculus **rHMŁ**, decide if φ is valid in **MŁ** or not. We repeat the steps to be performed: (i) We apply the analytic rules backwards, to arrive at quasiatomic r-hypersequents. (ii) To any non-trivial quasiatomic r-hypersequent, we apply the rules introduced in Lemmas 4.5 and 4.6 backwards, to arrive at quasiatomic r-hypersequents with specified Archimedean classes. (iii) For any quasiatomic r-hypersequents $\mathcal{H}$ with specified Archimedean classes, we check if $\mathcal{H}_b$ is valid. This can e.g. be done as described in [7, Theorem 4]. (iv) If step (iii) gives the negative answer, we write $\mathcal{H}$ as the disjoint union of quasiatomic r-hypersequents of type Łukasiewicz. (v) For each quasiatomic r-hypersequents of type Łukasiewicz, we check its validity. This can be done using linear programming methods [7, Theorem 2].

As might be expected, however, this procedure is certainly time-consuming. Step (i) produces a proof of a length which is exponential in the number of the connectives $\odot$ and $\rightarrow$. Step (ii) produces proofs of a length exponential in the sum of the square of the number of distinct atoms in each r-sequent. Steps (iii), (iv), (v), in contrast, can be done in polynomial time.

5 A hypersequent calculus for MΠ

In this section, we introduce an r-hypersequent calculus for **MΠ**, the logic of ordinal sums of the standard cancellative hoop. Our language can in this case be chosen smaller than in the case of **MŁ**. Naturally, there is no 0 constant; moreover, there is no analogue of the connective $\triangledown$ in **MΠ**. All propositions are in the language $\odot, \rightarrow$.

We proceed similarly as for **MŁ**, and the new calculus is called **rHMΠ**. In the absence of ∇, there are less analytic rules in **rHMΠ**; we just need the analogues of the **rHMŁ**-rules introducing $\odot$ and $\rightarrow$. However, the rules of **rHMΠ** concerning atomic r-hypersequents coincide with those of **rHMŁ** only partly; in particular, the above concept to separate cases according to the membership in Archimedean classes, is not applicable here.

Instead, we will propose the rule (S<) – see below –, which is a modified version of the rule (S') of **rHŁ**, which we mentioned in the proof of Lemma 4.13. Namely, we add to (S') further assumptions whose number is not fixed; it depends on the cardinality of a certain subset of the contained variables.

An r-hypersequent will be called *atomic* if it contains variables only. For an atomic r-hypersequent $\mathcal{H}$ and a set V of variables, we denote by $\left[\mathcal{H}\right]_V$ the r-hypersequent arising from $\mathcal{H}$ in the following way: (i) Delete all r-sequents not containing any variable in V; (ii) from every of the remaining r-sequents, remove the variables not in V.

Definition 5.1 The calculus **rHMΠ** consists of a set of *analytic rules*, and a set of *rules for atomic r-hypersequents*. The analytic rules of **rHMΠ** are the following:

$$(\odot\mathrm{l})\ \frac{\ldots \mid \Gamma, \alpha, \beta \leqq \Delta}{\ldots \mid \Gamma, \alpha \odot \beta \leqq \Delta} \qquad (\odot\mathrm{r})\ \frac{\ldots \mid \Gamma \leqq \Delta, \alpha, \beta}{\ldots \mid \Gamma \leqq \Delta, \alpha \odot \beta}$$

$$(\rightarrow\mathrm{l})\ \frac{\ldots \mid \Gamma \leqq \Delta \mid \Gamma, \beta \leqq \Delta, \alpha \qquad \ldots \mid \Gamma \leqq \Delta \mid \beta < \alpha}{\ldots \mid \Gamma, \alpha \rightarrow \beta \leqq \Delta}$$

$$(\rightarrow\mathrm{r})\ \frac{\ldots \mid \Gamma \leqq \Delta \qquad \ldots \mid \Gamma, \alpha \leqq \Delta, \beta \mid \alpha \leq \beta}{\ldots \mid \Gamma \leqq \Delta, \alpha \rightarrow \beta}$$

The rules of **rHMΠ** for atomic r-hypersequents are the following. Here, all assumptions are basic r-sequents, and lower-case Greek letters denote variables.

$$(\mathrm{A1})\ \emptyset \leq \emptyset \qquad (\mathrm{A2})\ \alpha \leq \alpha$$

$$(\mathrm{EW})\ \frac{\ldots}{\ldots \mid \Gamma \leqq \Delta} \qquad (\mathrm{EC})\ \frac{\ldots \mid \Gamma \leqq \Delta \mid \Gamma \leqq \Delta}{\ldots \mid \Gamma \leqq \Delta} \qquad (\mathrm{wl})\ \frac{\ldots \mid \Gamma \leqq \Delta}{\ldots \mid \Gamma, \alpha \leqq \Delta}$$

$$(\mathrm{M})\ \frac{\ldots \mid \Gamma_1 \leqq \Delta_1 \qquad \ldots \mid \Gamma_2 \leqq \Delta_2}{\ldots \mid \Gamma_1, \Gamma_2 \leqq \Delta_1, \Delta_2} \qquad (\mathrm{S})\ \frac{\ldots \mid \Gamma_1, \Gamma_2 \leqq \Delta_1, \Delta_2}{\ldots \mid \Gamma_1 \leqq \Delta_1 \mid \Gamma_2 \leqq \Delta_2}$$

(S<)

$$\frac{\Gamma_1, \Gamma_2 \leq \Delta_1, \Delta_2 \mid \mathcal{S} \qquad \overset{\text{for every non-empty set } \{\alpha_1, \ldots, \alpha_p\} \subseteq V:}{\emptyset \leq \alpha_1 \mid \ldots \mid \emptyset \leq \alpha_p \mid \left[\Gamma_2 < \Delta_2 \mid \mathcal{S}\right]_{\alpha_1, \ldots, \alpha_p}}}{\Gamma_1 \leq \Delta_1 \mid \Gamma_2 < \Delta_2 \mid \mathcal{S}},$$

where $\mathcal{S}$ is any r-hypersequent, and
V are the variables contained in Γ_2, Δ_2, or $\mathcal{S}$, but not in Γ_1 or Δ_1

A proposition α is said to be *provable* in **rHMΠ** if so is the sequent $\emptyset \leq \alpha$.

The notion of validity of r-hypersequents in **rHMΠ** involves exclusively the k-fold standard cancellative hoops T_k, where $k \geq 1$.

Definition 5.2 Let $v\colon \mathcal{P}_{\mathbf{M\Pi}} \to T_k$ be an evaluation of **MΠ**. We say that an r-sequent $\alpha_1, \ldots, \alpha_m \lesseqqgtr\ \beta_1, \ldots, \beta_n$, where $m, n \geq 0$, is *satisfied* by v if

$$v(\alpha_1) \cdot \ldots \cdot v(\alpha_m)\ \lneqq\ v(\beta_1) \cdot \ldots \cdot v(\beta_n); \tag{10}$$

here, $\lneqq$ denotes the order or strict order of T_k, respectively. Satisfaction by v and validity of r-sequents and r-hypersequents in **MΠ** are defined in analogy to Definition 4.2.

Validity will refer in this section always to **MΠ**.

Moreover, for simplicity, if v is an evaluation of **MΠ** and $\Gamma = \alpha_1, ..., \alpha_n$, we will write in the sequel $v(\Gamma) = v(\alpha_1) \cdot \ldots \cdot v(\alpha_n)$.

Lemma 5.3 *All rules of* **rHMΠ** *preserve validity. Moreover, all analytic rules of* **rHMΠ** *are invertible.*

Proof. It is obvious that the rules ($\odot$l) and ($\odot$r) are invertible. The fact that ($\to$l) and ($\to$r) are invertible, follows, just like in the case of **rHMŁ**, from Lemma 3.5(ii).

It is furthermore evident that (A1) and (A2) are satisfied by any evaluation and that (EW), (EC), (wl) preserve validity. (M) and (S) preserve validity by Lemma 3.7(i),(ii).

Let us now consider an instance of the rule (S<). Let v be any evaluation such that the assumptions of (S<) are satisfied by v. Assume furthermore that $\mathcal{S}$ is not satisfied by v, and let $\Gamma_1 \cup \Delta_1$ be non-empty.

Assume first that $\Gamma_1 \cup \Delta_1$ contains a variable α such that $v(\alpha) \preccurlyeq v(\Gamma_2)$. Then $v(\Gamma_1) \preccurlyeq v(\Gamma_2)$ or $v(\Delta_1) \preccurlyeq v(\Gamma_2)$; so by Lemma 3.7(iii), $v(\Gamma_1) \leq v(\Delta_1)$ or $v(\Gamma_2) < v(\Delta_2)$, that is, the conclusion of (S<) is satisfied by v.

Assume now that for all variables α in $\Gamma_1 \cup \Delta_1$, $v(\alpha) \succ v(\Gamma_2)$. Let $V = \{\alpha_1, \ldots, \alpha_p\}$ the set of those variables in (S<) such that $v(\alpha_i) < e$ and $v(\alpha_i) \preccurlyeq v(\Gamma_2)$ for $i = 1, ..., p$.

Let $\mathcal{A}$ be the assumption of (S<) associated to V. Then $\mathcal{A}$ is satisfied by v; none of the r-sequents $\emptyset \leq \alpha_i$, $i = 1, ..., p$, is satisfied by v; and none of the r-sequents in $\left[\mathcal{S}\right]_{\alpha_1, ..., \alpha_p}$ is satisfied by v. The last fact follows from Lemma 3.6, because any r-sequent in $\left[\mathcal{S}\right]_{\alpha_1, ..., \alpha_p}$ contains by definition at least one of the variables α_i, and we have $v(\alpha) = e$ or $v(\alpha) \succ v(\alpha_i)$ for any of the deleted variables α. So $\left[\Gamma_2 < \Delta_2\right]_{\alpha_1, ..., \alpha_p}$ is satisfied by v.

But again, $\Gamma_2 < \Delta_2$ contains at least one of the α_i's, and for any variable α different from $\alpha_1, \ldots, \alpha_p$, we have $v(\alpha) = e$ or $v(\alpha) \succ v(\Gamma_2)$. So by Lemma 3.6, $v(\Gamma_2) < v(\Delta_2)$, that is, $\Gamma_2 < \Delta_2$ is satisfied by v. This completes the proof that (S<) preserves validity. □

Lemma 5.4 *Applying the analytic rules of* **rHMΠ** *successively upwards to some r-hypersequent, terminates with atomic r-hypersequents.*

Proof. This is proved similarly as Lemma 4.4. □

Lemma 5.5 *Let $\mathcal{H}$ be an atomic r-hypersequent. If $\mathcal{H}$ is valid, then $\mathcal{H}$ is provable in* **rHMΠ**.

Proof. Let $\mathcal{H}$ be valid. Due to the presence of (EW), we may w.l.o.g. assume that any r-hypersequent arising from $\mathcal{H}$ by deleting an arbitrary r-sequent, is not valid. It is furthermore clear that there is an r-sequent in $\mathcal{H}$ with the relational symbol $\leq$. Finally, due to the presence of (A1), we may assume that $\emptyset \leq \emptyset$ is not in $\mathcal{H}$.

$\mathcal{H}$ is a valid r-hypersequent of **Ł** as well, that is, $\mathcal{H}$ is satisfied by all evaluations in T_1. From Motzkin's Transposition Theorem (see [19, Section 7.8], or the proof of [7, Theorem 8]), we conclude that there is an r-hypersequent

$$\mathcal{H}' = \Gamma_1 \leq \Delta_1 \ \mid \ \ldots \ \mid \Gamma_m \leq \Delta_m \ \mid \ \Phi_1 < \Psi_1 \ \mid \ \ldots \ \mid \ \Phi_n < \Psi_n \ \mid \ \mathcal{S} \tag{11}$$

such that (i) $\mathcal{H}'$ arises from $\mathcal{H}$ by multiplying some of its r-sequents, (ii) $m \geq 1$, and (iii) $\Gamma_1 \cup \ldots \cup \Gamma_m \cup \Phi_1 \cup \ldots \cup \Phi_n = \Delta_1 \cup \ldots \cup \Delta_m \cup \Psi_1 \cup \ldots \cup \Psi_n$.

We have to show that $\mathcal{H}'$ is provable in **rHMΠ**. If $n = 0$, that is, if $\Gamma_1 \leq \Delta_1 \ \mid \ \ldots \ \mid \ \Gamma_m \leq \Delta_m \ \mid \ \mathcal{S}$ is valid in **rHMΠ**, then we may prove $\mathcal{H}'$ from $\Gamma_1, \ldots, \Gamma_m \leq \Delta_1, \ldots, \Delta_m$ by (S) and (EW).

So assume $n \geq 1$. Apply (S<) such that

$$\Gamma_1, \ldots, \Gamma_m, \Phi_1, \ldots, \Phi_n \leq \Delta_1, \ldots, \Delta_m, \Psi_1, \ldots, \Psi_n \ \mid \ \mathcal{S}$$

is the first of its assumptions and

$$\Gamma_1, ..., \Gamma_m, \Phi_1, ..., \Phi_{n-1} \leq \Delta_1, ..., \Delta_m, \Psi_1, ..., \Psi_{n-1} \ \mid \ \Phi_n < \Psi_n \ \mid \ \mathcal{S}$$

is its conclusion. Repeating a similar step $n-1$ times gives

$$\mathcal{H}'' = \Gamma_1, \ldots, \Gamma_n \leq \Delta_1, \ldots, \Delta_n \ \mid \ \Phi_1 < \Psi_1 \ \mid \ \ldots \ \mid \ \Phi_n < \Psi_n \ \mid \ \mathcal{S},$$

from which $\mathcal{H}'$ is derivable by (S).

Now, all additional assumptions used in these steps are among those of the last application of (S<):

$$\emptyset \leq \alpha_1 \mid \ldots \mid \emptyset \leq \alpha_p \ \mid \ \left[\Phi_1 < \Psi_1 \mid \ \ldots \ \mid \Phi_n < \Psi_n \mid \mathcal{S}\right]_V, \tag{12}$$

where V is any non-empty subset of those variables in $\mathcal{H}'$ which are not contained in $\Gamma_1 \cup \ldots \cup \Gamma_m \cup \Delta_1 \cup \ldots \cup \Delta_m$. We claim that this r-hypersequent is valid. Indeed, let W contain the variables of $\mathcal{H}'$ not in V, and let v be an evaluation of W such that the r-hypersequent $\mathcal{H}'_W$ consisting of those r-sequents in $\mathcal{H}'$ whose variables are all in W, is not valid. Such an evaluation exists by our minimality assumption on $\mathcal{H}$.

Then extend v to an evaluation of all variables $V \cup W$ of $\mathcal{H}'$, choosing for the elements of V arbitrary values, but from Archimedean classes strictly below those in which the prior defined values are; if necessary, extend the range of v. Then the r-sequent satisfied by v must contain a variable in V, so in particular be one of $\Phi_1 < \Psi_1$, …, $\Phi_n < \Psi_n$ or in $\mathcal{S}$. It follows that the corresponding r-sequent in (12) and thus (12) itself is satisfied by v as well. The validity of (12) follows.

Furthermore, (12) contains, for any choice of the set V, at least one variable less than $\mathcal{H}$. So by an induction argument, the assertion follows. $\square$

The proof of this section's main assertion is complete.

Theorem 5.6 *The calculus* **rHMΠ** *is sound and complete for* **MΠ***: A proposition* α *is valid in* **MΠ** *if and only if* α *is provable in* **rHMΠ**.

Also in this case, it is possible to use the calculus to decide if a proposition of **MΠ** is valid in **MΠ** or not; we proceed as follows. (i) We apply the rules for $\odot$ and $\rightarrow$ backwards, to arrive at atomic r-hypersequents. (ii) For each atomic r-hypersequent $\mathcal{H}$, we find an r-hypersequent $\mathcal{H}'$ as specified in the proof of Lemma 5.5; see (11). (iii) We then write the proof of $\mathcal{H}'$ by means of (S) and (S<). For every additional assumption appearing due to a use of (S<), we return again to step (ii).

This method is, however, quite slow as a consequence of the special property of the rule (S<), to have a number of assumptions exponential in the number of variables not contained in Γ_1 or Δ_1.

6 Conclusion

We have formulated analytic proof systems for the logic **MŁ** of ordinal multiples of Łukasiewicz t-norms and for the logic **MΠ** of ordinal multiples of product t-norms on $(0, 1]$. These systems are based on r-hypersequents and fulfill a weakened form of the subformula property, such that the step-wise decomposition of a proposition leads to r-hypersequents containing no binary connectives. Because **MŁ** is a conservative extension of Hájek's Basic Logic **BL**, the results are applicable to this logic as well.

An effective search of a proof of a proposition is possible and, roughly speaking, done in three steps. First, the invertible rules for the logical connectives $\odot$ and $\rightarrow$ are applied backwards. The resulting quasiatomic or atomic r-hypersequents are then treated case-wise, according to the possibility that variables are in equal Archimedean classes or not. This second step is done within **rHMŁ** and **rHMΠ** in very different ways and in **rHMΠ**, it is actually intertwined with the final step. The final step is to check the validity of r-hypersequents w.r.t. Łukasiewicz logic, a task which can be done effectively by linear programming methods.

The presented calculi are not as elegant as those mentioned in the introduction for the standard extensions of **BL** and for **MTL**. In case of **rHMŁ**, it would be desirable to have a proof system in which quasiatomic r-hypersequents are derivable without the rule (Cut$\leq$/$>$). In case of **rHMΠ**, it would be desirable to go without the "superrule" (S<).

An analytic proof system for **BL** without the detour via **MŁ** and on the base of standard r-hypersequents, is still to be specified. We believe that such a system would be desirable also for the interpretational issue. For instance, in [11], the connection between analytic r-hypersequent systems on the one hand and dialogue-game based interpretation of fuzzy logics on the other hand was worked out. A compactly sized proof system for **BL** might support the aim to find a dialogue-game based characterization also for **BL**.

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