MTL-algebras arising from partially ordered groups

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Abstract

After dropping the implication-like operation and reversing the order, an MTL-algebra becomes a partially ordered structure $(L; \leq, \oplus, 0, 1)$ based on the single addition-like operation $\oplus$. Furthermore, $\oplus$ may be restricted to a partial, but cancellative addition $+$ without loss of information.

We deal in this paper with the case that the resulting partial algebra $(L; \leq, +, 0, 1)$ is embeddable into the positive cone of a partially ordered group. It turns out that most of the examples of MTL-algebras known at present can be represented in this way. So the systematization of a considerable class of MTL-algebras is achieved, and the methods of their construction are brought onto a common line.

1 Introduction

T-norm based fuzzy logics [13] represent an attempt to formalise reasoning about vague properties. By a property to be vague we mean that not for all objects of reference the property is either clearly true or clearly false. Fuzzy logics are in particular designed for situations in which natural-language expressions need to be modelled without artificial specifications about borderline cases. Applications where fuzzy logic has proved useful can for instance be found in medicine. In particular, we have recently formulated the inference mechanism of the medical expert system Cadiag-2 [1] in a fuzzy-logical framework [5].

In recent years, the algebraic aspects of fuzzy logics have been studied quite intensively, and the present paper is meant as a further contribution to this field. The algebraic semantics of a large family of fuzzy logics is based on subvarieties of the variety of residuated lattices [12]. In this paper, we consider the logic MTL which was introduced in [8] by F. Esteva and Ll. Godo. The standard models for MTL are left-continuous t-norm algebras: the real unit interval is used as the set of truth values and endowed with the natural order, an arbitrary left-continuous t-norm, the associated residual implication, and the constants 0 and 1. The variety generated by the left-continuous t-norm algebras are the bounded, integral, commutative, prelinear residuated lattices, called MTL-algebras.

Like residuated lattices in general, also MTL-algebras still resist a comprehensive analysis, and there do not seem to exist many results about their structure in general. Here, we intend to contribute partially to the characterisation of MTL-algebras; rather than considering the general case, we aim at systematising at least those MTL-algebras which are known at present. We have in particular in mind left-continuous t-norm algebras, various examples of which were collected in a paper of S. Jenei [18].

We will continue the lines of our previous work [24, 25]. These papers were focussed on BL-algebras, which are less general than MTL-algebras and whose structure is well-known [2]. In [24] we introduced weak effect algebras, which are partial algebras and a subclass of which can be identified with the BL-algebras. Under certain assumptions, we were able to perform an analysis of weak effect algebras analogously to the case of BL-algebras [25].

The basic idea on which this paper is based is the following. With every MTL-algebra, we may associate a partial algebra in a natural way (Section 2). For technical convenience, we first reverse the order. We then drop the difference-like operation and restrict the addition-like operation to the cases in which both summands are minimal in the sense that they cannot be made smaller without making the sum smaller. The resulting partially ordered structure $(L; \leq, +, 0, 1)$ is based on a single cancellative partial addition and bears all information of the original algebra.

In the next step, it would be natural to search for algebraic conditions implying that the partial algebra $(L; \leq, +, 0, 1)$ is embedabble into the positive cone of some partially ordered (po-) group. As far as the structure of these groups is known, we could investigate in this way our original algebras. However, the only condition which we know to be powerful enough, is the Riesz decomposition property, a property successfully applied to effect algebras and pseudoeffect algebras to obtain a representation by po-groups [21, 7]. However, in the present context this condition is too restrictive; for instance, the partial algebra associated to the nilpotent minimum t-norm would be excluded.

We have decided to go the opposite way; we shall examine MTL-algebras which do allow a representation by a po-group in the indicated way (Section 3). It turns out that this class is rather wide and contains in particular various examples from [18] (Section 4). All in all, we may say that we develop a systematic view on an important class of MTL-algebras, and we describe their construction in a uniform way.

2 The partial algebras associated to basic semihoops

This paper aims at characterising MTL-algebras; however, what we actually consider is slightly more general. MTL-algebras are bounded, commutative, integral, prelinear residuated lattices; we will drop here the boundedness condition. The resulting broader class of residuated lattices contains the so-called basic semihoops and corresponds to the falsehood-free version of MTL, called MTLH [9].

A *basic semihoop* is a structure $(S; \leq, \odot, \to, 1)$ such that (i) $(S; \leq)$ is a lattice with largest element 1; (ii) $(S; \odot, 1)$ is a commutative semigroup with neutral element 1; (iii) $\odot$ is isotone in both variables; (iv) for any a, b, $a \to b$ is maximal among all elements x such that $a \odot x \leq b$; and (v) for any a, b, we have $(a \to b) \vee (b \to a) = 1$. An *MTL-algebra* is a structure $(S; \leq, \odot, \to, 0, 1)$ such that $(S; \leq, \odot, \to, 1)$ is a basic semihoop whose least element is 0. Cf. [9, Lemma 3.13(a)].

We note that basic semihoops whose underlying order is complete coincide with the commutative, strictly two-sided quantales fulfilling the

so-called algebraic Strong de Morgan's law. For a general account on quantales, see [22]; moreover, the algebraic Strong de Morgan's law [20] is the prelinearity condition (v), see [22, Definition 4.3.3].

The prototypical examples of MTL-algebras are the standard models for the fuzzy logic MTL: left-continuous t-norm algebras. But it is quite remarkable, and possibly of significance for the debate about a proper interpretation of fuzzy logics, that MTL-algebras also naturally arise in other contexts, as the correspondence with quantales reveals. Namely, it is well-known that the set $\mathrm{Idl}(R)$ of left ideals of a ring R is naturally endowed with the structure of a quantale. If R is a commutative ring with 1, then the quantale $\mathrm{Idl}(R)$ is commutative and strictly two-sided. Let R be even a Dedekind domain; then $\mathrm{Idl}(R)$ fulfils the algebraic Strong de Morgan's law, and it follows that $\mathrm{Idl}(R)$ is actually a complete MTL-algebra.

For our purposes, it is reasonable to modify the definition of a basic semihoop in a purely technical way. Namely, we will work with the dual structures: we will reverse the order and rename constants and operations appropriately. Moreover, we will treat $\ominus$, which is the operation corresponding to $\rightarrow$, as a defined operation. These changes will make the relationship to po-groups more intuitive.

To avoid cumbersome terminology, we will rename the structures under study; in what follows, "dbs" is meant to remind of "dual basic semihoops".

Definition 2.1 A structure $(L; \leq, \oplus, 0)$ is called a *dbs-algebra* if the following holds.

(T1) $(L; \leq, 0)$ is a lattice with smallest element 0.

(T2) $(L; \oplus, 0)$ is a commutative semigroup with neutral element 0.

(T3) $\oplus$ is isotone in both variables.

(T4) For any a and b, there is a smallest element x such that $a \oplus x \geq b$. This element will be denoted by $b \ominus a$.

(T5) For a, b, let $a \ominus b$ and $b \ominus a$ as specified by (T4). Then $(a \ominus b) \wedge (b \ominus a) = 0$.

A dbs-algebra is called *bounded* if there is a largest element, denoted then by 1. By a *dbs-chain*, we mean a dbs-algebra whose order is linear.

For a quick reference of notions, we list the one-to-one correspondences:

dbs-algebras	—	basic semihoops (commutative, integral, prelinear residuated lattices)
bounded dbs-algebras	—	MTL-algebras
dbs-chains	—	linearly ordered basic semihoops
bounded dbs-chains	—	linearly ordered MTL-algebras

This article is devoted exclusively to dbs-chains. This is motivated by the fact that dbs-algebras are subdirect products of linearly ordered ones [9, Lemma 3.10]. Note that in the case of a linear order, axiom (T5) is redundant.

Next we shall see that dbs-algebras are in a one-to-one correspondence with certain partial algebras.

Definition 2.2 Let $(L; \leq, \oplus, 0)$ be a dbs-algebra. Define the partial binary operation $+$ as follows: for $a, b \in L$, let $a + b = a \oplus b$ if a is the smallest element x such that $x \oplus b = a \oplus b$ and b is the smallest element y such that $a \oplus y = a \oplus b$; else let $a + b$ be undefined. Then $+$ is called the *partial addition belonging to* $\oplus$, and $(L; \leq, +, 0)$ is called the *partial algebra associated to* L.

Proposition 2.3 *Let $(L; \leq, \oplus, 0)$ be a dbs-algebra, and let $+$ be the partial addition belonging to $\oplus$. Then $\oplus$ can be reobtained from $+$ by*

$$a \oplus b = \max \{a' + b' \colon\ a' \leq a \text{ and } b' \leq b \text{ such that } a' + b' \text{ is defined}\}$$

for any $a, b \in L$.

Proof. Let $a, b \in L$ and $c = a \oplus b$. If $a' \leq a$ and $b' \leq b$ such that $a' + b'$ exists, clearly $a' + b' \leq c$.

On the other hand, let $a' = c \ominus b$; then a' is the smallest element fulfilling $a' \oplus b \geq c$. Because then in particular $a' \leq a$, we also have $a' \oplus b \leq c$, whence $a' \oplus b = c$.

Similarly, we set $b' = c \ominus a'$, so that $a' \oplus b' = c$. In this sum, b' is minimal by construction. Furthermore, $c \ominus b' \leq a' = c \ominus b \leq c \ominus b'$, that is, $c \ominus b' = a'$, whence also a' is minimal. So we get $a \oplus b = a' + b'$. $\square$

To give an impression of the nature of the partial algebras with which we have to do here, we collect some of their characteristic properties. In other words, if we had to axiomatise these algebras, we would find among the axioms at least the following ones.

Proposition 2.4 *Let $(L; \leq, \oplus, 0)$ be a dbs-algebra, and let $(L; \leq, +, 0)$ be the partial algebra associated to L. Then the following holds.*

(P1) *$(L; \leq, 0)$ is a lattice with smallest element 0.*

(P2) *$+$ is a partial binary operation such that we have for $a, b, c \in L$:*

- (i) *$a + b$ is defined if and only if $b + a$ is defined, in which case both elements are equal.*
- (ii) *If $(a + b) + c$ and $a + (b + c)$ are both defined, then these elements are equal.*
- (iii) *$a + 0$ is defined and equals a.*

(P3) *If $a+c$ and $b+c$ are both defined, then $a \leq b$ exactly if $a+c \leq b+c$.*

These properties strongly remind of generalised effect algebras; see e.g. [6]. However, there are two important differences. In contrast to generalised effect algebras, the partial order of the algebras derived from dbs-algebras need not be determined by the partial addition: there may be elements a and b such that $a \leq b$, but no c such that $a + c = b$.

Moreover, there are, in principle, several different kinds of associativity for partial algebras; for this topic see [3]. Generalised effect algebras fulfil the strongest form: $(a + b) + c$ is defined if and only if $a + (b + c)$ is defined, in which case both these elements coincide. In the present context, however, we have associativity only in the weak form (P2)(ii).

We finally mention another closely related kind of algebra: the weak effect algebras, with which we deal in [24, 25] and which arise from

BL-algebras. The lack of the property that the addition determines the partial order, is shared by weak effect algebras. But weak effect algebras do fulfil the strong version of associativity. In the case that they arise from BL-algebras, even the Riesz decomposition property holds [25].

3 Basic semihoops constructed from linearly ordered groups

We have seen that every dbs-algebra (alias basic semihoop alias commutative, integral, prelinear residuated lattice) gives rise to an algebra based on a partial addition and that the transition to the partial algebra means no loss of information. However, apart from the basic properties (P1)–(P3) in Proposition 2.4, it is far from easy to characterise the partial algebras arising this way. In this section, we shall investigate those among them which can be embedded into some po-group.

We shall wonder which kind of a substructures of a po-group are associated to a dbs-algebra in the sense of Definition 2.2. The basic idea is as follows. We choose a subset of the positive cone of a po-group; we restrict the group operation to this subset; and the obtained partial addition is re-extended to a total operation in the way shown in Proposition 2.3. The question is then under which conditions the result is a dbs-algebra.

In the sequel, we will mark total group operations by a hat, like for instance $\hat{+}$, so as to distinguish them from the partial operations.

Definition 3.1 Let $(G; \leq, \hat{+}, 0)$ be a po-group. Let $L \subseteq G^+$ such that $0 \in L$, and let $+$ be a partial binary operation on L such that for all $a, b \in L$ (i) $a + b$, whenever defined for some $a, b \in L$, coincides with $a \,\hat{+}\, b$ and (ii) $a + 0$ is always defined. Moreover, let $\leq$ be the restriction of the partial order of G^+ to L. Then we call $(L; \leq, +, 0)$ a *partial substructure* of $(G^+; \leq, \hat{+}, 0)$.

Moreover, considering $(L; \leq, +, 0)$, assume that

$$a \oplus b = \max \{a' + b' \colon\ a' \leq a,\ b' \leq b,\ a' + b' \text{ is defined}\} \qquad (1)$$

exists for all $a, b \in L$. Then we say that $+$ is *extendible* to $\oplus$, and we call $\oplus$ the *total addition* on L belonging to $+$. Moreover, $(L; \leq, \oplus, 0)$ is called the *total algebra associated to* $(L; \leq, +, 0)$.

A dbs-algebra which is the total algebra associated to the partial substructure of some po-group's positive cone is called *po-group representable*.

We will see in Section 4 that the dbs-chains which are duals of left-continuous t-norm algebras are po-group representable in numerous cases.

Definition 3.1 shows how a dbs-algebra might arise from a partial algebra; and Definition 2.2 shows how a partial algebra is associated to a dbs-algebra. Performing successively these two steps should lead us to the original algebra.

Proposition 3.2 *Let G be a po-group, and let $(L; \leq, +, 0)$ be a partial substructure of G^+. Let $+$ be extendible, and assume that $(L; \leq, \oplus, 0)$ is a dbs-algebra. Then the partial algebra associated to $(L; \leq, \oplus, 0)$ is again $(L; \leq, +, 0)$.*

Proof. In this proof, $+$ denotes always the partial addition of the original algebra $(L; \leq, +, 0)$.

Let a and b be minimal in the sum $c = a \oplus b$ in the sense of Definition 2.2. According to (1), let $a' \leq a$ and $b' \leq b$ such that $c = a' + b'$. So $a \oplus b = a' + b' = a' \oplus b'$, and it follows $a' = a$ and $b' = b$. So $c = a + b$ is defined.

Conversely, let $c = a + b$ be defined. Then $c = a \oplus b$, and in this sum, a and b are obviously minimal. □

We will now explicitely describe ways of constructing partial substructures of the positive cone of a po-group such that the associated total algebra is a dbs-algebra. There are various possibilities to do so, and we do not try to give the most general solution; the one presented here is technically sufficiently involved. For the sake of simplicity, we would actually prefer less general methods than those offered below. However,

decreasing generality would very likely exclude some of the examples from Section 4.

We will consider linearly ordered (l.o.) groups only. These groups are described by the Hahn embedding theorem; see e.g. [11]. This theorem, although not explicitly stated, provided actually the main idea for this paper.

In what follows, if R is a po-group and $u \in R^+$, $R[0,u]$ denotes the interval of R between 0 and u.

Definition 3.3 Let $1 \leq \lambda \leq \omega$, and for every $i < \lambda$ let $(R_i; \leq, \hat{+}_i, 0)$ be a subgroup of the additive group of reals, endowed with the natural order. Let $R = \Gamma_{i<\lambda} R_i$ be the lexicographic product of the R_i, with addition $\hat{+}$. We will refer to $(R^+; \leq, \hat{+}, 0)$ as a *standard l.o. group cone*.

Let $0 \leq j < \lambda$. For $a \in R$, $a_j \in R_j$ denotes the j-th component of a. Furthermore, we denote by $R_{<j}$ the set of $a \in R$ such that $a_i = 0$ for $i \geq j$. For any $a \in R$, we denote by $a_{<j}$ the element of $R_{<j}$ arising from a by setting all components with indices $\geq j$ to 0. Similarly, we define $R_{\leq j}$ and $a_{\leq j}$. Finally, for some $L \subseteq R$, we define $L_{<j} = L \cap R_{<j}$ and $L_{\leq j} = L \cap R_{\leq j}$.

For a partial substructure $(L; \leq, +, 0)$ of $(R^+; \leq, \hat{+}, 0)$, let us define the conditions (Bas1)–(Bas3) as follows.

(Bas1) The set $L \subseteq R^+$ has the following properties.

(i) $0 \in L$.

(ii) Let $0 \leq j < \lambda$, and for every $a \in L_{<j}$, let

$$R_j(a) = \{r \in R_j\colon\ a^{(r)} \in L\},$$

where $a^{(r)} \in R_{\leq j}$ is defined by $a^{(r)}_{<j} = a_{<j}$ and $a^{(r)}_j = r$. Then $R_j(0)$ is one of $R_j[0,M]$ for some $M \geq 0$, or R_j^+; if $a > 0$ and a is the maximal element in $L_{<j}$, then $R_j(a)$ is one of R_j^- or $R_j[m,0]$ for $m \leq 0$; else $R_j(a)$ is one of R_j^-, R_j, $R_j[0,M]$ for $M \geq 0$, or R_j^+.

(iii) For any $a \in R$, $a \in L$ if and only if $a_{<j} \in L$ for all j.

(Bas2) Let $\bar{R}_j = \cup_{a \in L_{<j}} R_j(a)$ for $0 \leq j < \lambda$. There are partial binary operations $+_j$ on $\bar{R}_j$ which determine the partial addition $+$ on L as follows: for $a, b \in L$, $a + b$ is defined iff $a_j +_j b_j$ is defined for all j, in which case $(a + b)_j = a_j +_j b_j$ for all j.

For each j, the operation $+_j$ has the following properties. For $r, s \in \bar{R}_j$, (i) if $r +_j s$ is defined, it equals $r \hat{+}_j s$ (i.e. the sum of r and s in the group R_j); (ii) if $r +_j s$ is defined, so is $s +_j r$; (iii) $r +_j 0$ is always defined.

If, for $r, s \in \bar{R}_j$,

$$\{r' +_j s' \colon\ r', s' \in \bar{R}_j,\ r' \leq r,\ s' \leq s\}$$

has a maximal element, we will denote it by $r \oplus_j s$.

(Bas3) Let $0 \leq j < \lambda$ and $a, b, c \in L_{<j}$.

(i) Let $a + b = c$. If $R_j(a)$ is upper bounded, then $R_j(b)$ is lower bounded, $R_j(c)$ is upper bounded, and we have

$$\max R_j(a) \hat{+} \min R_j(b) \geq \max R_j(c). \tag{2}$$

(ii) Let $a \leq c$. Then

$$\inf R_j(a) \geq \inf R_j(c). \tag{3}$$

Now, to give rise to a dbs-chain, the partial substructure of a standard l.o. group cone must fulfil three conditions: the partial operation $+$ must be extendible to $\oplus$ according to (1); the total addition $\oplus$ must be associative; and $\oplus$ must fulfil the condition (T4).

We will give for these three conditions sufficient criteria. We will assume (Bas1)–(Bas3), and we will, roughly speaking, require that the partial algebras $(\bar{R}_j, \leq, +_j, 0)$, $1 \leq j < \lambda$, have the respective property. In addition, we will make further assumptions on upper and lower bounds of the ranges $R_j(a)$ for certain $a \in L_{<j}$; and a restricted form of the Riesz decomposition property will be postulated for $(\bar{R}_j, \leq, +_j, 0)$.

The following auxiliary definition serves to simplify later formulations.

Definition 3.4 Let $(L; \leq, +, 0)$ be a partial substructure of a standard l.o. group cone $((\Gamma_{i<\lambda} R_i)^+; \leq, \hat{+}, 0)$ such that (Bas1)–(Bas3) hold. Let $0 \leq j < \lambda$ and $a, b \in L_{\leq j}$ such that $a + b$ is defined. Then we say that a_j is in the sum $a + b$ *at an upper definitional limit* if there is an $e \in R_j^+ \setminus \{0\}$ such that for every $a' \in R_{\leq j}$ fulfilling $a_{<j} = a'_{<j}$ and $a_j < a'_j \leq a_j \hat{+}_j e$, either $a' \notin L$ or else $a' + b$ is undefined. Under an analogous condition, we say that a_j is in the sum $a + b$ *at a lower definitional limit.*

Lemma 3.5 *Let $(L; \leq, +, 0)$ be a partial substructure of a standard l.o. group cone $((\Gamma_{i<\lambda} R_i)^+; \leq, \hat{+}, 0)$ such that* (Bas1)–(Bas3) *hold. Then $+$ is extendible to $\oplus$ if the following holds.*

(Ext1) *Let $0 \leq j < \lambda$ and $a, b, c \in L_{<j}$ such that $a + b = c$. Then for all $r \in R_j(a)$ and $s \in R_j(b)$, $r \oplus_j s$ exists.*

(Ext2) *Let $0 \leq j \leq \lambda$ and $a \in L_{<j}$. Assume $\sup R_j(a) = M < \infty$. Then for all $r, s \in R_j$ such that $r +_j s \geq M$, there are $r' \leq r$ and $s' \leq s$ in R_j such that $r' +_j s' = M$.*

(Ext3) *Let $0 \leq j < \lambda$ and $a, b, c \in L_{\leq j}$ such that $a + b = c$. Let a_j be in the sum $a + b$ at an upper definitional limit. Then $R_{j+1}(c)$ is upper bounded.*

In this case, $c = a \oplus b$ for some $a, b \in L$ is determined as follows. Let $j \leq \lambda$ largest such that $a_{<j} + b_{<j}$ is defined. Then we have $c_i = a_i +_i b_i$ for $i < j$, $c_j = (a_j \oplus_j b_j) \wedge \sup R_j(c_{<j})$, and $c_i = \max R_i(c_{<i})$ for all $i > j$.

Proof. Let (Ext1)–(Ext3) hold, and let $a, b \in L$ such that $a + b$ is undefined. We have to specify $c = a \oplus b$ according to (1), that is, find $a' \leq a$ and $b' \leq b$ such that $a' + b'$ exists and is as large as possible.

Let $j < \lambda$ be largest such that $a_{<j} + b_{<j}$ is defined, and set $c_{<j} = a_{<j} + b_{<j}$. Furthermore, according to (Ext1), $a_j \oplus_j b_j$ exists; let $c_j = (a_j \oplus_j b_j) \wedge \sup R_j(c_{<j})$. We claim that then, in accordance with (1),

$$\begin{aligned} c_j \;=\; & \max \{r +_j s\colon\ r \in R_j(a_{<j}) \text{ and } r \leq a_j; \\ & s \in R_j(b_{<j}) \text{ and } s \leq b_j; \\ & r +_j s \text{ is defined and in } R_j(c_{<j})\}. \end{aligned} \tag{4}$$

Indeed, there are $r', s' \in R_j$ such that $r' \leq a_j$, $s' \leq b_j$ and $a_j \oplus_j b_j = r' +_j s'$. We claim that $r' \in R_j(a_{<j})$ and $s' \in R_j(b_{<j})$. Indeed, if $a_j < 0$ and $R_j(a_{<j})$ is lower bounded by m_j, then $r' \in R_j(a_{<j})$, because then $a_{<j}$ is maximal in $L_{<j}$ by (Bas1)(ii) and consequently, by (3), m_j is the minimal element of $\bar{R}_j$, the set on which $+_j$ operates. The other cases are analogous or obvious.

If now $a_j \oplus_j b_j \in R_j(c_{<j})$, we get in (4) the maximum for $r = r'$, $s = s'$. Otherwise, if $R_j(c_{<j})$ is lower bounded by m_j, then $a_j \oplus_j b_j$ cannot be strictly smaller than m_j because, again, m_j is then the minimal element of $\bar{R}_j$. So if $a_j \oplus_j b_j \notin R_j(c_{<j})$, then $a_j \oplus_j b_j$ exceeds the upper bound M_j of $R_j(c_{<j})$. In this case, there are by (Ext2) $r \leq r'$ in $R_j(a_{<j})$ and $s \leq s'$ in $R_j(b_{<j})$ which add up to M_j. So (4) is proved.

Let now r, s be a maximising pair in (4). Let $a' \leq a$ be such that $a'_{<j} = a_{<j}$ and $a'_j = r$, and let $b' \leq b$ be such that $b'_{<j} = b_{<j}$ and $b'_j = s$; then $a'_{\leq j} + b'_{\leq j} = c_{\leq j}$. Note that either $a'_j < a_j$ or $b'_j < b_j$; so by (Ext3), $R_{j+1}(c_{<(j+1)})$ has an upper bound M_{j+1}. It follows from (2) that $M_{j+1} \in R_{j+1}(a'_{<(j+1)})$; so we may set $a'_{j+1} = M_{j+1}$, $b'_{j+1} = 0$, $c_{j+1} = M_{j+1}$. We may apply now (Ext3) again to set a'_{j+2} and c_{j+2} to the upper bound of $R_{j+2}(c_{<(j+2)})$ and b'_{j+2} to 0; and so forth. Then $c = a' + b'$ is the maximal element being the sum of a pair $a' \leq a$ and $b' \leq b$, that is, $c = a \oplus b$. □

Lemma 3.6 *Let $(L; \leq, +, 0)$ be a partial substructure of a standard l.o. group cone $((\Gamma_{i<\lambda} R_i)^+; \leq, \hat{+}, 0)$ such that* (Bas1)–(Bas3) *and* (Ext1)–(Ext3) *hold. Then $\oplus$ is associative if the following holds.*

(Ass) *Let $0 \leq j \leq \lambda$, let $a, b, c \in L_{<j}$ such that $(a + b) + c$ is defined, and let $a_j \in R_j(a)$, $b_j \in R_j(b)$, $c_j \in R_j(c)$. Then $(a_j \oplus_j b_j) \oplus_j c_j = a_j \oplus_j (b_j \oplus_j c_j)$.*

Proof. Let $a, b, c \in L$, $d = (a \oplus b) \oplus c$, and $e = a \oplus (b \oplus c)$. We have to show $d = e$.

Assume first that $(a + b) + c$ exists; so $d_j = (a_j +_j b_j) +_j c_j$ for every j. Let $j \geq 0$, and assume that $e_{<j} = a_{<j} + (b_{<j} + c_{<j})$ is defined; note that this holds trivially in case $j = 0$. It follows $d_{<j} = e_{<j}$. By (Ass), also $a_j +_j (b_j +_j c_j)$ is defined in R_j. Assume that $b_{\leq j} + c_{\leq j}$ is

undefined; then $b_j +_j c_j \notin R_j(b_{<j} + c_{<j})$, which by Lemma 3.5 means that $b_j +_j c_j$ is strictly greater than the upper bound of $R_j(b_{<j} + c_{<j})$. This in turn means by (2) that also $a_j +_j (b_j +_j c_j)$ is outside $R(e_{<j}) = R(d_{<j})$, a contradiction. So $b_{\leq j} + c_{\leq j}$ is defined and consequently also $a_{\leq j} + (b_{\leq j} + c_{\leq j})$. It follows $d_{\leq j} = e_{\leq j}$. By induction, we conclude $d = e$.

Assume next that $(a + b) + c$ does not exist, and let j be smallest such that $(a_{\leq j} + b_{\leq j}) + c_{\leq j}$ does not exist. Then we argue as in the preceeding paragraph to conclude $d_{<j} = (a_{<j} + b_{<j}) + c_{<j} = a_{<j} + (b_{<j} + c_{<j}) = e_{<j}$. Furthermore, by (Ass), $(a_j \oplus_j b_j) \oplus_j c_j = a_j \oplus_j (b_j \oplus_j c_j)$. If then $a_j \oplus_j b_j \notin R_j(a_{<j} + b_{<j})$, $a_j \oplus_j b_j$ exceeds the upper bound M_j of $R_j(a_{<j} + b_{<j})$, and by (2) $M_j \oplus_j c_j \geq \max R_j(d_{<j})$, that is, $d_j = \max R_j(d_{<j})$; furthermore, if $b_j \oplus_j c_j$ is in $R_j(b_{<j} + c_{<j})$ or not, in both cases we have $e_j = \max R_j(e_{<j}) = \max R_j(d_{<j}) = d_j$. If, on the other hand, $a_j \oplus_j b_j \in R_j(a_{<j} + b_{<j})$ and $b_j \oplus_j c_j \in R_j(b_{<j} + c_{<j})$, we again have $d_j = e_j$. By Lemma 3.5, $d_{j+1} = e_{j+1}, \ldots$ are, respectively, the maximal values of the allowed range. □

Lemma 3.7 *Let $(L; \leq, +, 0)$ be a partial substructure of a standard l.o. group cone $((\Gamma_{i<\lambda} R_i)^+; \leq, \hat{+}, 0)$ such that* (Bas1)–(Bas2) *and* (Ext1)–(Ext3) *hold. The total addition $\oplus$ belonging to $+$ fulfils* (T4) *if the following holds.*

(Res1) *Let $0 \leq j < \lambda$ and $a, b, c \in L_{<j}$ such that $a + b = c$. Then for all $r \in R_j(a)$ and $t \in R_j(c)$, there is a smallest $s \in R_j(b)$ such that $r \oplus_j s \geq t$.*

(Res2) *Let $0 \leq j < \lambda$ and $a, b \in L_{\leq j}$ such that $a + b$ is defined. W.r.t. the sum $a + b$, let either a_j be at an upper definitional limit or b_j at a lower definitional limit. Then $R_{j+1}(b)$ is lower bounded.*

Proof. Let $a, c \in L$ such that $a < c$. We have to show that there is smallest $b \in L$ such that $a \oplus b \geq c$. If there is a $b \in L$ such that $a + b$ exists and equals c, we are done.

Assume on the contrary that $j < \lambda$ is the largest index such that a $d \in L_{<j}$ fulfilling $a_{<j} + d = c_{<j}$ exists; let $b \in L$ be such that $b_{<j} = d$.

By (Res1), there is a smallest $s \in R_j(b_{<j})$ such that $a_j \oplus_j s \geq c_j$; let b_j be this element. So $b_{\leq j}$ is the minimal element such that $a_{\leq j} \oplus b_{\leq j} \geq c_{\leq j}$.

Now, there is an $r \leq a_j$ such that $r +_j b_j = a_j \oplus_j b_j$; note that $r \in R_j(a_{<j})$. Let $a' \in L_{\leq j}$ be such that $a'_{<j} = a_{<j}$ and $a'_j = r$. In case $a'_j = a_j$, $a'_j +_j b_j > c_j$ and b_j is in the sum $a' + b$ at a lower definitional limit. In case $a'_j < a_j$, a'_j is in the sum $a' + b$ at an upper definitional limit. So by (Res2), $R_{j+1}(b_{<(j+1)})$ has a lower bound, to which value we set b_{j+1}.

Again by (Res2), we may choose successively also $b_{j+2}, \ldots$ as the minimal elements of the respective range. Then b is as required. □

We may now summarise all preceeding lemmas.

Theorem 3.8 *Let $(L; \leq, +, 0)$ be a partial substructure of a standard l.o. group cone. Assume that L fulfils* (Bas1)–(Bas3)*,* (Ext1)–(Ext3)*,* (Ass)*, and* (Res1)–(Res2)*. Then the total algebra belonging to L is a dbs-chain.*

So in particular, if L is as specified in Theorem 3.8 and moreover bounded, then L gives rise to a linearly ordered MTL-algebra.

4 Examples

We will now present a list of examples of left-continuous t-norms which are constructable by use of Theorem 3.8 from an l.o. group.

It seems easiest to proceed according to the following scheme. We start from some specific left-continuous t-norm $\odot_\star \colon [0,1]^2 \to [0,1]$, distinguished by some symbol $\star$. The corresponding bounded basic semihoop is then $([0,1]; \leq, \odot_\star, \to_\star, 1)$, where $\leq$ is the natural ordering of the reals and $\to_\star$ is the residuum belonging to $\odot_\star$. We next pass to the corresponding dbs-chain $([0,1]; \leq, \oplus_\star, 0)$; for this, we simply have to calculate the t-conorm $a \oplus_\star b = 1 - ((1-a) \odot_\star (1-b))$, $a, b \in [0,1]$. Then, we pass to the associated partial algebra $([0,1]; \leq, +_\star, 0)$; for this, we have to determine the partial addition $+_\star$ on L from $\oplus_\star$

according Definition 2.2. Finally, we define a partial substructure of some standard l.o. group cone which is isomorphic to $([0,1];\leq,+_\star,0)$.

To save space, we will make use of the symmetry of the operations; when $a \odot_\star b$ is defined for certain choices of a and b, we will assume that $b \odot_\star a$ is defined as well. The same applies to $\oplus_\star$ and $+_\star$. Moreover, in case of $+_\star$, we will treat *only* those cases in which both summands are non-zero; $a +_\star 0 = a$ is defined for any element a anyhow. Furthermore, the partial operation on the partial substructure of an l.o. group cone is understood as being defined whenever it can be performed, unless explicitely otherwise noticed. Note finally that in this section, we denote the total operations on $\mathbb{R}$ simply by $+$, $-$, $\cdot$, without a hat.

We will begin with the three basic continuous t-norms. In these cases, we are led to well-known constructions.

Example 4.1 Let $\odot_L$ be the Łukasiewicz t-norm: for $a, b \in [0,1]$, let

$$a \odot_L b = (a + b - 1) \vee 0.$$

Then $([0,1];\leq,\odot_L,1)$ is a linearly ordered basic semihoop. Let $([0,1];\leq,\oplus_L,0)$ be the corresponding dbs-chain and $([0,1];\leq,+_L,0)$ the partial algebra associated to the latter; we then have

$$\begin{aligned} a \oplus_L b &= (a+b) \wedge 1; \\ a +_L b &= a + b \text{ in case } a + b \leq 1. \end{aligned}$$

For the po-group representation, we let $R = \mathbb{R}$, $L = R[0,1]$. Then $(L;\leq,+,0)$ is a partial substructure of the standard l.o. group cone R^+, which is evidently isomorphic to $([0,1];\leq,+_L,0)$.

Next, let $\odot_P$ be the product t-norm: for $a, b \in [0,1]$, let

$$a \odot_P b = a \cdot b.$$

Then we have

$$\begin{aligned} a \oplus_P b &= a + b - a \cdot b; \\ a +_P b &= a + b - a \cdot b \text{ in case } a, b < 1. \end{aligned}$$

For the po-group representation, we let $R_0 = \mathbb{N}$, $R_1 = \mathbb{R}$, $R = R_0 \times_{\text{lex}} R_1$, and

$$L = \{(a,b) \in R\colon\ a = 0,\ b \geq 0 \text{ or } a = 1,\ b = 0\}.$$

Finally, let $\odot_G$ be the Gödel t-norm: for $a, b \in [0,1]$, let

$$a \odot_G b = a \wedge b.$$

Then we have $\oplus_G = \vee$, and $+_G$ is not defined for any pair of non-zero elements.

For the po-group representation, we let $R = \mathbb{R}$, $L = R[0,1]$, and we do not define $+$ for any pair of non-zero positive elements.

Now, general continuous t-norms are ordinal sums of the standard t-norms treated above. This motivates us to demonstrate how we may form ordinal sums within our framework. Definition 3.3 implies one restriction for this construction: Either there are only finitely many summands, or countably infinitely many ones ordered like the naturals.

We note that Definition 3.3 could actually generalised to the case of any linear order. This would show that actually any continuous t-norm is po-group representable.

Proposition 4.2 *Let $1 \leq \mu \leq \omega$, and for every $k < \mu$ let $i_k \in \mathbb{N}$ such that $0 = i_0 < i_1 < \ldots$. For every $k < \mu$, let $(L_k; \leq, +_k, 0)$ be a partial substructure of a standard l.o. group cone $((\Gamma_{i_k \leq i < i_{k+1}} R_i)^+; \leq, \hat{+}, 0)$. Assume that the total algebras belonging to $(L_k; \leq, +_k, 0)$, $k < \mu$ are dbs-algebras.*

Furthermore, let $\lambda = \sup_{k<\mu} i_{k+1}$ and

$$L = \{a \in (\Gamma_{i<\lambda} R_i)^+ \colon \text{ for some } k < \mu,\ a_i = 0 \text{ for } i < i_k \text{ or } i \geq i_{k+1}\}.$$

Let $\phi_k \colon L_k \to L$ be for $k < \mu$ be the natural embedding, and define $a + b$ in L if $a, b \in \phi_k(L_k)$ for some k and $\phi_k^{-1}(a) +_k \phi_k^{-1}(b)$ are defined in L_k. Then the total algebra belonging to $(L; \leq, +, 0)$ is dbs-chain.

We now turn to non-continuous left-continuous t-norms, first to the annihilation construction [10, 15].

Example 4.3 Let $\odot_n$ be the nilpotent minimum t-norm: for $a, b \in [0,1]$, let

$$a \odot_n b = \begin{cases} a \wedge b & \text{if } a + b > 1, \\ 0 & \text{else.} \end{cases}$$

Then we have

$$\begin{aligned} a \oplus_n b &= \begin{cases} a \vee b & \text{if } a+b<1, \\ 1 & \text{else;} \end{cases} \\ a +_n b &= 1 \quad \text{in case } a+b=1. \end{aligned}$$

For the po-group representation, we let $R = \mathbb{R}$, $L = R[0,1]$, and for $a, b > 0$, we define $a + b$ in case $a + b = 1$.

Next, we consider the t-norm $\odot_J$: for $a, b \in [0,1]$, let

$$a \odot_J b = \begin{cases} a \wedge b & \text{if } a+b>1 \text{ and } a \le \frac{1}{3} \text{ or } a > \frac{2}{3}, \\ a+b-\frac{2}{3} & \text{if } a+b>1 \text{ and } \frac{1}{3} < a, b \le \frac{2}{3}, \\ 0 & \text{if } a+b \le 1. \end{cases}$$

Then we have

$$\begin{aligned} a \oplus_J b &= \begin{cases} a \vee b & \text{if } a+b<1 \text{ and } a < \frac{1}{3} \text{ or } a \ge \frac{2}{3}, \\ a+b-\frac{1}{3} & \text{if } a+b<1 \text{ and } \frac{1}{3} \le a, b < \frac{2}{3}, \\ 1 & \text{if } a+b \ge 1; \end{cases} \\ a +_J b &= \begin{cases} a+b-\frac{1}{3} & \text{if } a+b<1 \text{ and } \frac{1}{3} < a, b < \frac{2}{3}, \\ 1 & \text{if } a+b=1. \end{cases} \end{aligned}$$

For the po-group representation, we let $R_0 = R_1 = \mathbb{R}$, $R = R_0 \times_{\text{lex}} R_1$, we define $a +_1 b$ for $a, b \in R_1$ if $a + b = 0$, and we set

$$\begin{aligned} L = \{(a,b) \in R^+\colon \quad & a = 0,\ 0 \le b \le 1 \quad \text{or } 0 < a < 1,\ b = 0 \\ & \text{or } a = 1,\ -1 \le b \le 0\}. \end{aligned}$$

Next, we turn to the rotation construction [16].

Example 4.4 Let $\odot_{rP}$ the rotated product t-norm: for $a, b \in [0,1]$, let

$$a \odot_{rP} b = \begin{cases} 2ab - a - b + 1 & \text{if } a, b > \frac{1}{2}, \\ \frac{a+b-1}{2a-1} & \text{if } a > \frac{1}{2},\ b \le \frac{1}{2}, \text{ and } a+b>1, \\ 0 & \text{if } a+b \le 1. \end{cases}$$

Then we have

$$\begin{aligned} a \oplus_{rP} b &= \begin{cases} a+b-2ab & \text{if } a, b < \frac{1}{2}, \\ \frac{b-a}{1-2a} & \text{if } a < \frac{1}{2},\ b \ge \frac{1}{2}, \text{ and } a+b<1, \\ 1 & \text{if } a+b \ge 1; \end{cases} \\ a +_{rP} b &= \begin{cases} a+b-2ab & \text{if } a, b < \frac{1}{2}, \\ \frac{b-a}{1-2a} & \text{if } a < \frac{1}{2},\ b > \frac{1}{2}, \text{ and } a+b \le 1, \\ 1 & \text{if } a = b = \frac{1}{2}. \end{cases} \end{aligned}$$

For the po-group representation, we let $R_0 = \mathbb{N}$, $R_1 = \mathbb{R}$, $R = R_0 \times_{\text{lex}} R_1$, and

$$L = \{(a,b) \in R^+\colon\ a = 0,\ b \geq 0 \text{ or } a = 1,\ b = 0 \text{ or } a = 2,\ b \leq 0\}.$$

The isomorphism between $([0,1]; \leq, +_{rP}, 0)$ and $(L; \leq, +, 0)$ is given by $\varphi\colon\ [0,1] \to L$, where for $a \in [0,1]$

$$\varphi(a) = \begin{cases} (0, -\ln(1-2a)) & \text{if } a < \frac{1}{2}, \\ (1,0) & \text{if } a = \frac{1}{2}, \\ (2, \ln(2a-1)) & \text{if } a > \frac{1}{2}. \end{cases}$$

We now consider one case of the rotation-annihilation construction [17].

Example 4.5 Let $\odot_{raLL}$ be the rotation-annihilation of two Łukasiewicz t-norms: for $a, b \in [0,1]$, let

$$a \odot_{raLL} b = \begin{cases} a+b-1 & \text{if } a,b > \frac{2}{3} \text{ and } a+b > \frac{5}{3} \\ & \text{or } a \leq \frac{1}{3},\ b > \frac{2}{3} \text{ and } a+b > 1, \\ \frac{2}{3} & \text{if } a,b > \frac{2}{3} \text{ and } a+b \leq \frac{5}{3}, \\ a+b-\frac{2}{3} & \text{if } \frac{1}{3} < a,b \leq \frac{2}{3} \text{ and } a+b > 1, \\ a & \text{if } \frac{1}{3} < a \leq \frac{2}{3},\ b > \frac{2}{3}, \\ 0 & \text{if } a+b \leq 1. \end{cases}$$

Then we have

$$a \oplus_{raLL} b \ = \ \begin{cases} a+b & \text{if } a,b < \frac{1}{3} \text{ and } a+b < \frac{1}{3} \\ & \text{or } a < \frac{1}{3},\ b \geq \frac{2}{3} \text{ and } a+b < 1, \\ \frac{1}{3} & \text{if } a,b < \frac{1}{3} \text{ and } a+b \geq \frac{1}{3}, \\ a+b-\frac{1}{3} & \text{if } \frac{1}{3} \leq a,b < \frac{2}{3} \text{ and } a+b < 1, \\ a & \text{if } \frac{1}{3} \leq a < \frac{2}{3} \text{ and } b < \frac{1}{3}, \\ 1 & \text{if } a+b \geq 1; \end{cases}$$

$$a +_{raLL} b \ = \ \begin{cases} a+b & \text{if } a+b < \frac{1}{3} \text{ or } a < \frac{1}{3},\ b \geq \frac{2}{3},\ a+b < 1, \\ a+b-\frac{1}{3} & \text{if } \frac{1}{3} < a,b < \frac{2}{3} \text{ and } a+b < 1, \\ 1 & \text{if } a+b = 1. \end{cases}$$

For the po-group representation, we let $R_0 = R_1 = \mathbb{R}$, $R = R_0 \times_{\text{lex}} R_1$, and

$$L = \{(a,b) \in R^+\colon \quad a = 0,\ 0 \leq b \leq 1 \quad \text{or } 0 < a < 1,\ b = 0 \quad \text{or } a = 1,\ -1 \leq b \leq 0\}.$$

We conclude by giving two more examples of t-norms; in these cases, however, we will just describe their associated partial algebras.

Example 4.6 Let $R_0 = R_1 = \mathbb{N}$, $R_2 = \mathbb{R}$ and $R = R_0 \times_{\text{lex}} R_1 \times_{\text{lex}} R_2$; let

$$L = \{(a, b, c) \in R^+ \colon\ a = 0,\ b \geq 0,\ c \geq 0 \text{ or } a = 1,\ b = c = 0\}.$$

Then from $(L; \leq, +, 0)$, the t-norm suggested by P. Hájek in [14] arises. Finally, let $R_0 = R_1 = \ldots = \mathbb{N}$, $R = \Gamma_{i<\omega} R_i$, and

$$L = \{(a_i)_i \in R^+ \colon\ a_0 = 0 \text{ or } a_0 = 1,\ a_1 = a_2 = \ldots = 0\}.$$

This gives rise to the t-norm constructed by D. Hliněná in [23]; cf. also [4].

These two last examples give rise to a comparison of our approach to the method developed by S. Jenei and F. Montagna in [19]. Expressed in our terminology, in [19] standard l.o. group cones $(\Gamma_{i<\lambda} R_i)^+$ are considered, where R_i, $i < \lambda$, are the integers with the possible exception of the last index, for which it may be the reals. The construction is furthermore based on any monoidal operations on the R_i, not necessarily on the group operation like in our case. On the other hand, restrictions of the base set as proposed in this paper are not supposed in [19].

In both cases of Example 4.6, when removing the largest element from L, we have the full positive cone of an l.o. group, and the partial addition is a total operation on it. In such a situation, our construction does not differ from [19]; we just work in a different formal setting.

5 Conclusion

Any basic semihoop, so in particular any MTL-algebra, can be identified with an algebra $(L; \leq, +, 0)$ whose crucial property is the cancellativity of the partial operation $+$. We have provided numerous examples in which L can be isomorphically embedded into the positive cone of a partially ordered group. Furthermore, with reference to the group we have given sufficient conditions that this type of representation exists.

The most important open question is how to characterize algebraically those basic semihoops which allow a description by means of a po-group in the indicated way.

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