

Algorithm for the generation of the Archimedean, finite, negative tomonoids

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Abstract—We study Archimedean, finite, negative totally ordered monoids. We describe an algorithm which generate the structures of this type in a step-wise fashion. Our approach benefits from the level set representation of monoids and is inspired by web geometry.

I. INTRODUCTION

Monoids are important structures in many fields of mathematics, including logic, computer science, or the theory of languages. In fact, in non-classical logic, monoids are often used to interpret the conjunction. The monoids occurring in this context are then typically endowed with a translation-invariant partial order.

In this contribution, we focus on a special class of partially ordered monoids: we assume that the order is total; that the monoidal identity coincides with the top element; and finiteness. Our interest comes from the field of residuated lattices [GJKO]. Under the additional assumption of commutativity, the structures that we consider can in fact be identified with finite MTL-algebras [EsGo]; MTL-algebras are in turn the algebraic counterpart of the fuzzy logic MTL.

We utilize, as we call it, the *level set approach*, which is inspired by the field of web geometry [Acz], [BIBo]. A tomonoid can be represented by its level sets and associativity then corresponds to the so-called Reidemeister condition. The level-set approach has been applied already to triangular norms [PeSa1] and has been utilized to make a significant progress in some open problems on convex combinations of triangular norms [PeSa2].

Furthermore, our previous paper [PeVe] exploits this approach for the discussion of finite, negative tomonoids. In particular, we explain how to construct the elementary extensions of such tomonoids. An elementary extension is by one element larger and the identification of its two smallest elements leads back to the tomonoid we have started from. Starting from the one-element tomonoid, the successive formation of elementary extensions leads to any given finite, negative tomonoid.

In the present paper, we focus on the algorithmic aspects of the construction described in [PeVe]. We restrict, to this end, to the case that the tomonoids under consideration are Archimedean. For proofs and further details of the underlying theory, we refer to [PeVe].

II. BASIC NOTIONS

Definition 2.1: A monoid is an algebra $(S; \odot, 1)$ of type $\langle 2, 0 \rangle$ such that, for any $a, b, c \in S$,

$$(T1) \quad (a \odot b) \odot c = a \odot (b \odot c),$$

$$(T2) \quad a \odot 1 = 1 \odot a = a.$$

A total (linear) order \leq on a monoid S is called *compatible* if, for every $a, b, c \in S$,

$$(T3) \quad a \leq b \text{ implies } a \odot c \leq b \odot c \text{ and } c \odot a \leq c \odot b.$$

In this case, we call $(S; \leq, \odot, 1)$ a *totally ordered monoid*, or a *tomonoid* for short. Further, we call S *negative* if 1 is the top element and we call S *commutative* if, for every $a, b \in S$,

$$(T4) \quad a \odot b = b \odot a.$$

In this paper, we are exclusively interested in finite, negative, tomonoids, abbreviated “*f. n. tomonoids*”. Let us remark that, in contrast to [EKMMW], we do not assume commutativity, although we deal also with this case.

The smallest tomonoid, called the *trivial tomonoid*, is the one that consists of the monoidal identity 1 alone. Tomonoids with at least two elements are called *non-trivial*.

A negative tomonoid is called *Archimedean* if, for every $x, y \in S \setminus \{1\}$ such that $x \leq y$, there is an $n \in \mathbb{N}$ such that $y^n \leq x$. Here, we define

$$y^n = \underbrace{y \odot y \odot \dots \odot y}_{n\text{-times}}.$$

We note that negative tomonoids with at most two elements are trivially Archimedean.

III. LEVEL-SET VIEW ON TOMONOIDS

In this section we introduce the representation of tomonoids by level sets. Let $\odot: S \times S \rightarrow S$ be a binary operation on a totally ordered set S and let \sim be a binary relation on $S \times S$ such that, for $a, b, c, d \in S$,

$$(a, b) \sim (c, d) \quad \text{iff} \quad a \odot b = c \odot d.$$

We can see that \sim is an equivalence relation and that it partitions $S \times S$ into those subsets of pairs that are mapped

by \odot to equal values. When recovering \odot from \sim we need to know which equivalence class is associated with which value of S . But this is easy if \odot possesses a neutral element $1 \in S$. In such a case each class contains exactly one pair of the form $(1, a)$ and one pair of the form $(a, 1)$. Furthermore, for every $a \in S$ there is exactly one class containing the pairs $(1, a)$ and $(a, 1)$. This gives us a one-to-one mapping between the equivalence classes and the elements from S .

Thus, a tomonoid $(S; \leq, \odot, 1)$ can be characterized by the totally ordered set $(S; \leq)$, the equivalence relation \sim defining a partition on $S \times S$, and the designated element 1. This simple idea gives to our hand a tool that is geometric in nature and, in contrast with the graph of binary operations, gets along with two dimensions only.

Definition 3.1: Let $(S; \leq, \odot, 1)$ be a tomonoid. For two pairs $(a, b), (c, d) \in S \times S$ we define

$$(a, b) \sim (c, d) \quad \text{iff} \quad a \odot b = c \odot d$$

and we call \sim the *level equivalence* associated with S .

Definition 3.2: We denote by \trianglelefteq the componentwise order on $S \times S$ for some totally ordered set S , i.e., for every $a, b, c, d \in S$, we put

$$(a, b) \trianglelefteq (c, d) \quad \text{iff} \quad a \leq b \quad \text{and} \quad c \leq d.$$

Definition 3.3: Let $(S; \leq)$ be a totally ordered set, let $1 \in S$, and let \sim be an equivalence relation on $S \times S$ such that the following holds.

- (P1) For every $a, b, c, d, e \in S$,
 $(a, b) \sim (1, d)$ and $(b, c) \sim (1, e)$ imply $(d, c) \sim (a, e)$.
- (P2) For every $a, b \in S$ there is exactly one $c \in S$ such that
 $(a, b) \sim (1, c) \sim (c, 1)$.
- (P3) For every $a, b, c, d, a', b', c', d' \in S$,
 $(a, b) \sim (a', b') \trianglelefteq (c, d) \sim (c', d') \trianglelefteq (a, b)$
implies
 $(a, b) \sim (c, d)$.

Then we call $(S, \leq, 1, \sim)$ a *tomonoid partition*.

Proposition 3.4: [PeVe] Let $(S; \leq, \odot, 1)$ be a tomonoid and let \sim be its level equivalence. Then $(S, \leq, 1, \sim)$ is a tomonoid partition.

Proposition 3.5: [PeVe] Let $(S, \leq, 1, \sim)$ be a tomonoid partition. For every $a, b \in S$, let

$$a \odot b := \text{the unique } c \text{ such that } (a, b) \sim (1, c) \sim (c, 1).$$

Then $(S; \leq, \odot, 1)$ is the unique tomonoid such that $(S, \leq, 1, \sim)$ is its associated tomonoid partition.

We conclude from Proposition 3.4 and Proposition 3.5 that tomonoids and tomonoid partitions are in a one-to-one correspondence.

For a tomonoid S , the level equivalence \sim partitions the set $S \times S$ into as many equivalence classes as there are elements of S . In fact, in view of (P2), the classes and the elements of S are in an one-to-one correspondence. Moreover, the equivalence

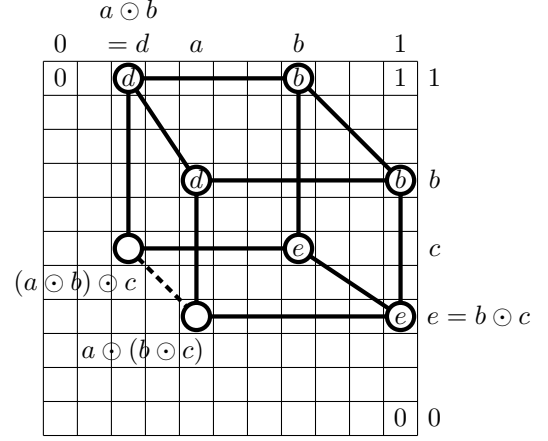


Fig. 1. Illustration of Property (P1) of a tomonoid partition. As depicted, this property corresponds with the associativity of the tomonoid.

classes inherit under this correspondence the total order from S .

Property (P1) is related to the associativity of the tomonoid and has the following geometric interpretation illustrated in Figure 1. On the square $S \times S$, consider two rectangles such that one hits the upper edge and the other one hits the right edge. Assume that the upper left, upper right, and lower right vertices of these rectangles are in the same equivalence classes, respectively. Then, by (P1), also the remaining lower left vertices are elements of the same equivalence class. The described property corresponds with the *Reidemeister condition* known web geometry [Acz], [BIBo].

In the sequel we will use the following simplified notation. Instead of $(a, b) \sim (1, c)$, or equivalently $(a, b) \sim (c, 1)$, we will simply write $(a, b) \sim c$.

Finally, let us specify the tomonoid partitions that correspond to additional properties of tomonoids.

Observation 3.6: Let $(S; \leq, \odot, 1)$ be a tomonoid and let $(S, \leq, 1, \sim)$ be its associated tomonoid partition.

- S is commutative if, and only if, the equivalence classes of \sim are “mirrored by the diagonal”, i.e., if $(a, b) \sim (b, a)$ for every $a, b \in S$.
- S is negative if, and only if, for every $c \in S$ the \sim -class of c is contained in $\{(a, b) \in S \times S \mid a, b \geq c\}$.
- S is finite if, and only if, \sim is an equivalence relation on a finite set.

IV. REES QUOTIENTS AND ELEMENTARY EXTENSIONS

For a finite totally ordered set, we will refer to the least element as the *zero*, to the second smallest element as its *atom*, and to the second greatest element as its *coatom*.

We now introduce the notion of an elementary extension of a f. n. tomonoid [PeVe]. A f. n. tomonoid \bar{S} will be an elementary extension of a f. n. tomonoid S such that the cardinality of \bar{S} is greater by one and S is a quotient of \bar{S} . Furthermore, this quotient will be the simplest non-trivial Rees

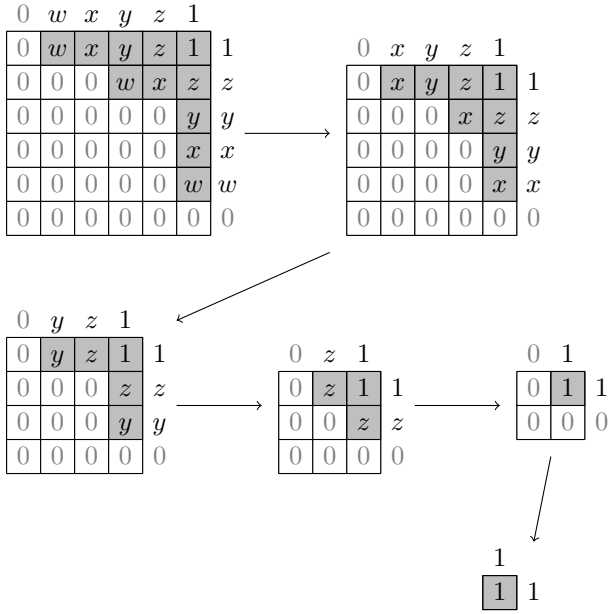


Fig. 2. Elementary quotients a f. n. tomonoid of size 6. The second tomonoid is the elementary quotient of the first one, the third is the elementary quotient of the second one, etc. Finally, we reach the trivial monoid.

quotient, called the elementary quotient of a f. n. tomonoid: it arises from merging the zero with the atom.

See an illustration in Figure 3 which depicts, successively, all the elementary quotients of a given f. n. tomonoid. As we can see in the tomonoid partition picture, the elementary quotient arises by “cutting off” the column and the row indexed by the zero and, further, the zero class and the atom class are joined to a new class which is evaluated to the new smallest element.

Exact definitions now follow. For unexplained general notions of see, e.g., [Gri].

Definition 4.1: Let $(S; \leq, \odot, 1)$ be a tomonoid. A *tomonoid congruence* on S is an equivalence relation \approx on S such that

- (i) \approx is a congruence of S as a monoid and
- (ii) each equivalence class is convex.

The operation induced by \odot on the quotient $\langle S \rangle$ we denote again by \odot . For $a, b \in S$, we define $\langle a \rangle \leq \langle b \rangle$ if $a \approx b$ or $a < b$.

We may observe that $(\langle S \rangle; \leq, \odot, \langle 1 \rangle)$ is a tomonoid again and we call $\langle S \rangle$ the *tomonoid quotient* w.r.t. \approx . Clearly, the properties of finiteness, negativity, and commutativity are preserved by this procedure. What follows is a definition of the Rees congruence which is commonly used for semi-groups [How].

Lemma 4.2: Let $(S; \leq, \odot, 1)$ be a negative tomonoid and let $q \in S$. For $a, b \in S$, let $a \approx_q b$ if $a = b$ or $a, b \leq q$. Then \approx_q is a tomonoid congruence.

Definition 4.3: Let $(S; \leq, \odot, 1)$ be a f. n. tomonoid and let $q \in S$. We call \approx_q , as defined in Lemma 4.2, the *Rees*

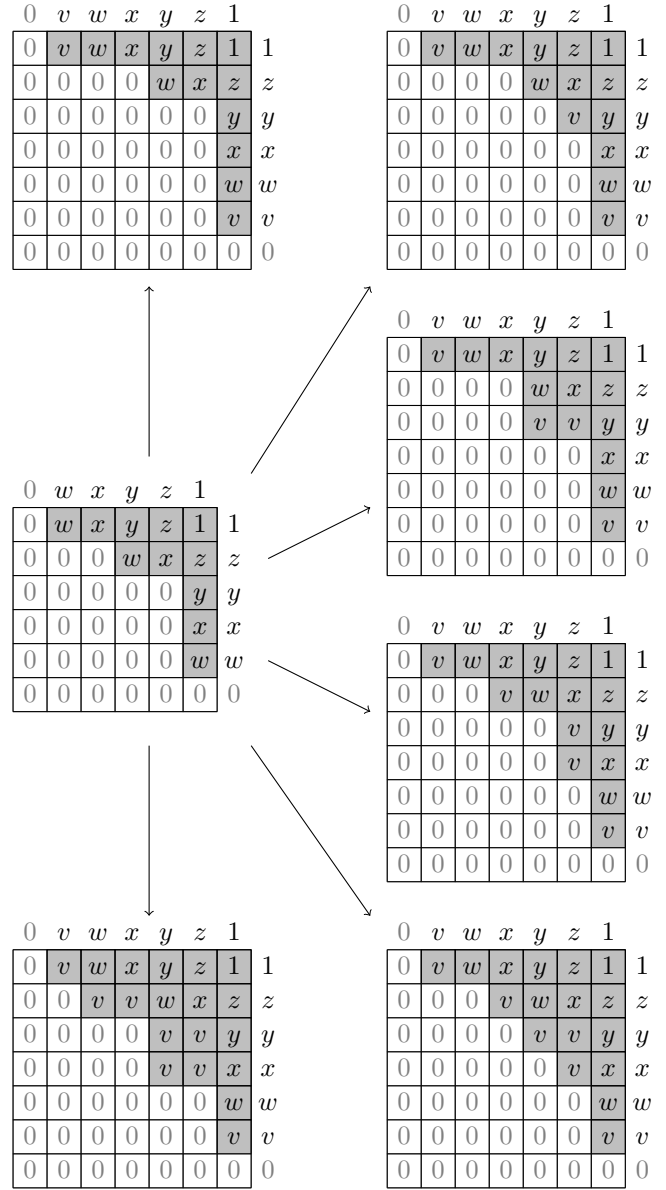


Fig. 3. All the elementary extensions of a f. n. tomonoid of size 6.

congruence w.r.t. q . Furthermore, we denote the quotient by S/q and we call it the *Rees quotient* of S w.r.t. q .

Further, let \bar{S} be a non-trivial f. n. tomonoid and let α be the atom of \bar{S} . We call the Rees quotient S of \bar{S} w.r.t. α the *elementary quotient* of \bar{S} and, conversely, \bar{S} an *elementary extension* of S .

V. ARCHIMEDEAN ELEMENTARY EXTENSIONS

We now turn to the main problem of determining the elementary extensions of an Archimedean f. n. tomonoid which we present in the form of an algorithm.

Algorithm 5.1:

Input: $(S; \leq, 1, \sim)$... tomonoid partition of an Archimedean f. n. tomonoid

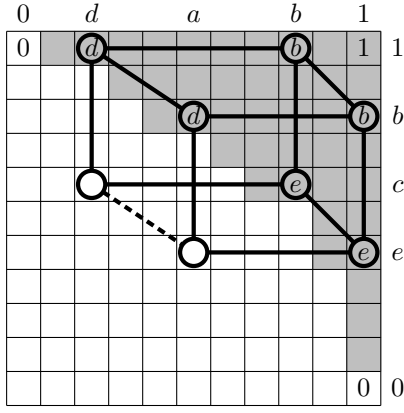


Fig. 4. Illustration of Step 8 of Algorithm 5.1.

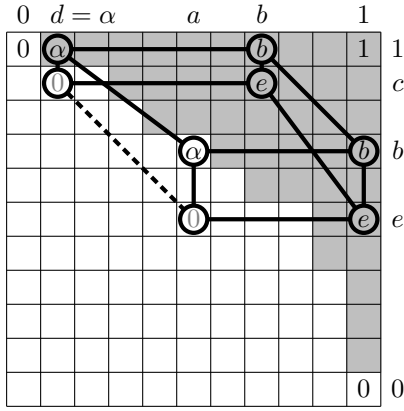


Fig. 5. Illustration of Step 9 and Step 10 of Algorithm 5.1.

Output: $(\bar{S}, \leq, 1, \sim) \dots$ elementary extension of $(S, \leq, 1, \sim)$

- 1) Let $\bar{S} = S \cup \{\bar{0}\}$, where $\bar{0}$ is a new element.
- 2) Endow \bar{S} with the total order that extends the total order on S such that $\bar{0} < a$ for every $a \in S$.
- 3) Let α be the atom of \bar{S} (i.e., the least element of S).
- 4) Let κ be the coatom of \bar{S} .
- 5) Let

$$P = \{(a, b) \in \bar{S} \times \bar{S} \mid \text{there is } c > \alpha \text{ such that } (a, b) \sim c\}.$$
- 6) Define a binary relation \sim on $\bar{S} \times \bar{S}$ such that

$$(a, b) \sim (c, d) \text{ iff } (a, b) \sim (c, d) \sim e \text{ for some } e \in \bar{S} \setminus \{\bar{0}, \alpha\}$$
- 7) For every $a \in \bar{S} \setminus \{\bar{0}\}$:
 - define $(a, \bar{0}) \sim (a, \alpha) \sim (\bar{0}, a) \sim (\alpha, a) \sim \bar{0}$.
- 8) For every $(a, b), (b, c) \in P$:
 - let $d \in \bar{S}$ be such that $(a, b) \sim d$,
 - let $e \in \bar{S}$ be such that $(b, c) \sim e$,
 - define $(a, e) \sim (d, c)$.

9) For every $a \in \bar{S} \setminus \{\bar{0}, \alpha, 1\}$:

- let $b \in \bar{S}$ be the highest element such that $(a, b) \notin P$,
- let $e \in \bar{S}$ be such that $(b, \kappa) \sim e$,
- for every $(x, y) \in \bar{S} \times \bar{S}$ such that $(x, y) \leq (a, e)$:
 - define $(x, y) \sim \bar{0}$.

10) For every $a \in \bar{S} \setminus \{\bar{0}, \alpha, 1\}$:

- let $b \in \bar{S}$ be the highest element such that $(b, a) \notin P$,
- let $e \in \bar{S}$ be such that $(\kappa, b) \sim e$,
- for every $(x, y) \in \bar{S} \times \bar{S}$ such that $(x, y) \leq (e, a)$:
 - define $(x, y) \sim \bar{0}$.

11) Let

$$R = \{(a, b) \in \bar{S} \times \bar{S} \mid \text{there is no } c \in \bar{S} \text{ such that } (a, b) \sim c\}.$$

12) Relate each pair in R with either $\bar{0}$ or α regarding monotonicity and \sim .

Remark 5.2: In Step 8, the pairs, where $a = 1, b = 1$, or $c = 1$, may be omitted as they bring no new information to \sim . Step 8, Step 9, and Step 10 are illustrated by Figure 4 and Figure 5, respectively.

All the steps of the algorithm run in polynomial time except for Step 12 which has exponential complexity. This is, however, related to the fact that also the number of the results increases exponentially with the number of the pairs in R .

Theorem 5.3: Let $(S, \leq, 1, \sim)$ be an Archimedean f. n. tomonoid partition. The partition $(\bar{S}, \leq, 1, \sim)$ given by Algorithm 5.1 is an Archimedean elementary extension of $(S, \leq, 1, \sim)$ and, moreover, all its Archimedean elementary extension arise in this way.

See [PeVe] for a proof of this theorem.

VI. THE COMMUTATIVE CASE

Under the assumption of commutativity, looking for an elementary extension of an Archimedean f. n. tomonoid can be performed analogously to Algorithm 5.1 with the following differences.

Step 9 and Step 10 can be merged to one single step:

- For every $a \in \bar{S} \setminus \{\bar{0}, \alpha, 1\}$:
 - let $b \in \bar{S}$ be the highest element such that $(a, b) \notin P$,
 - let $e \in \bar{S}$ be such that $(b, \kappa) \sim e$,
 - for every $(x, y) \in \bar{S} \times \bar{S}$ such that $(x, y) \leq (a, e)$:
 - define $(x, y) \sim (y, x) \sim \bar{0}$.

Step 12 needs to be changed in the way that when a pair $(a, b) \in R$ is related to $\bar{0}$ -class or with α -class then the reversed pair (b, a) has to be related to the same class, as well.

VII. LOWER BOUNDS

In this section we want to show that the number of Archimedean f. n. tomonoid increases rapidly with the number of the elements and that it is lower-bounded by a function given by a binomial coefficient in the general non-commutative case and by an exponential function in the commutative case.

Definition 7.1: A finite tomonoid $(S; \leq, \odot, 1)$ with the least element 0 is called *drastic* if, for every $a, b \in S$, $a, b \neq 1$, we have $a \odot b = 0$.

The following lemma is an easy observation.

Lemma 7.2: Let $(S; \leq)$ be a finite, totally ordered set with the least element 0, the greatest element 1, and the atom α . Let \odot be a binary operation on S defined, for $a, b \in S$, by

$$\begin{aligned} a \odot b &= a && \text{if } b = 1, \\ a \odot b &= b && \text{if } a = 1, \\ a \odot b &= 0 && \text{if } a \in \{0, \alpha\} \text{ and } b < 1 \\ a \odot b &= 0 && \text{if } b \in \{0, \alpha\} \text{ and } a < 1, \\ a \odot b &\in \{0, \alpha\} && \text{otherwise.} \end{aligned}$$

Then S is an Archimedean f. n. monoid.

If, moreover, the two following conditions are fulfilled:

- (i) for every $a \in S$ there is $c \in S$ such that, for every $b \in S$, we have $a \odot b = 0$ if and only if $b \leq c$,
- (ii) for every $b \in S$ there is $c \in S$ such that, for every $a \in S$, we have $a \odot b = 0$ if and only if $a \leq c$,

then \odot is compatible with \leq and S is an Archimedean f. n. tomonoid.

Proof: We start with the first part of the theorem. Clearly, 1 is a neutral element of S . Thus we only need to prove that the associativity equation $a \odot (b \odot c) = (a \odot b) \odot c$ holds for every $a, b, c \in S$. If any of a, b, c is equal to 1 then the equation is trivially satisfied. If this is not the case then it can be easily checked that both sides of the equation are equal to 0.

We are going to prove (T3) as stated in Definition 2.1. Take $a, b, c \in S$ such that $a \leq b$. If $c = 1$ then both $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$ hold trivially. If this is not the case then, according to (i), we have exactly one of the following cases:

- $c \odot a = 0$ and $c \odot b = 0$,
- $c \odot a = 0$ and $c \odot b = \alpha$,
- $c \odot a = \alpha$ and $c \odot b = \alpha$

which implies $c \odot a \leq c \odot b$. In an analogous manner (ii) implies $a \odot c \leq b \odot c$. ■

In an inspiration with this lemma we can see that, in order to obtain all the elementary extensions of a drastic tomonoid, we simply need to discover in how many ways we can place a block of zero elements and a block of atom elements to the multiplication table regarding only the monotonicity (and, eventually, also the commutativity). This brings the following statement.

Proposition 7.3: Let $(S; \leq, \odot, 1)$ be a drastic f. n. tomonoid of size $n+1$, $n \in \mathbb{N}$. There are $\binom{2n}{n}$ elementary

extensions of S that are Archimedean f. n. tomonoids. There are 2^n elementary extensions of S that are commutative, Archimedean f. n. tomonoids.

Proof: Let $(\bar{S}; \leq, \bar{\odot}, 1)$ be an elementary extension of S ; let $\bar{0}$ be the zero of \bar{S} , let α be the atom of \bar{S} . Since \bar{S} is supposed to be Archimedean, we necessarily have $(a, b) \sim 0$ if $a \in \{0, \alpha\}$ and $b < 1$ or if $b \in \{0, \alpha\}$ and $a < 1$ (cf. with Lemma 7.2). Denote $I = \bar{S} \setminus \{0, \alpha, 1\}$ and note that the cardinality of I is n . Since S is drastic, we need, in order to obtain an Archimedean f. n. tomonoid \bar{S} , to make a partition dividing $I \times I$ to Z and A such that

$$\begin{aligned} (a, b) &\sim 0 && \text{if } (a, b) \in Z, \\ (a, b) &\sim \alpha && \text{if } (a, b) \in A \end{aligned}$$

for every $(a, b) \in I \times I$. According to Lemma 7.2, regardless of the partition, \bar{S} will be an Archimedean f. n. monoid. Thus the only thing we need to take into account is the compatibility of $\bar{\odot}$ with \leq . This will be satisfied if and only if Z and A are separated by a border that consists of n vertical and n horizontal segments. There are $\binom{2n}{n}$ such borders.

If \bar{S} is, moreover, supposed to be commutative then the border separating Z and A needs to be symmetric according to the diagonal. A half of this border consists of n segments which are either vertical or horizontal. There are 2^n such borders. ■

Thus, as we can see, the number of f. n. tomonoids of a given size is lower-bounded by a binomial coefficient in the non-commutative Archimedean case and by an exponential function in the commutative Archimedean case.

VIII. COMPARISON WITH BRUTE-FORCE METHODS

We give here a short comparison of our level-set based method with two brute-force methods. For this purpose, we have generated all the Archimedean f. n. tomonoids up to the size 10 and measured the times in which the methods have run. The times of running the level-set based method are, as expected, shorter which illustrates the following table:

size	number	lvl	bru	sup
3	1	0.0003	0.0004	0.0003
4	2	0.0007	0.0005	0.0008
5	8	0.0038	0.0029	0.0328
6	44	0.0235	0.0329	77.4722
7	333	0.2430	5.7076	-
8	3543	3.4303	32848.5	-
9	54954	67.7673	-	-
10	1297705	2864.3895	-	-

Column “size” represents the sizes of the generated tomonoids and “number” represents their numbers. The used methods are the following.

Column “lvl” shows the times to run the algorithm based on our *level-set method*. In order to obtain all the tomonoids of the given size, first, the trivial tomonoid is created, then all its elementary extensions are computed, then the extensions of the extensions are computed, and so on. This way we are creating a tree of tomonoids until the level of the required size is reached.

Column “bru” represents, what we call, the *brute force method*. The idea is similar to the previous case and a tree of elementary extensions is created. However, to obtain all the extensions of a given tomonoid, we simply iterate through all the possible evaluations of the complement of the set P (see Step 5 of Algorithm 5.1) and we discard those cases that fail the tests on associativity, monotonicity, and Archimedeanity.

Column “sup” contains times of, what we call, the *super-brute-force method*. In this case, in order to obtain all the tomonoids of the given size, we generate all the possible multiplication tables and test them on the requirements on associativity, monotonicity, and Archimedeanity.

The algorithm has been implemented in Python and run on a personal computer with 1,3 GHz Intel Core i5 processor and 4GB of memory—this is evidently not a computer dedicated for such a task and thus the absolute times are not very significant. However, what might give an illustration are the ratios of the times of the different methods which are as follows:

size	bru/lvl	sup/lvl
3	1.3	1.0
4	0.7	1.1
5	0.7	8.6
6	1.4	3296.7
7	23.5	-
8	9576	-
9	-	-
10	-	-

IX. CONCLUSION

An algorithm to give Archimedean elementary extensions and commutative Archimedean elementary extensions of a finite, negative, tomonoid has been presented. The authors, at the moment, are working also on the algorithm for the general, non-Archimedean, case.

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