International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems

Completing fuzzy if-then rule bases by means of smoothing splines

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Received 24.2.2005
Revised 20.7.2005

A fuzzy if-then rule base may be viewed as a partial function between universes of fuzzy sets. For the construction of a fuzzy inference module, this partial function needs to be extended to a total one.

Here, we propose a new method how to do so, making use of the method of smoothing splines. To this end, we identify the fuzzy sets with elements of a finite-dimensional real parameter space in an approximate way, using Perfilieva's fuzzy transforms. We then determine a function between two such parameter spaces by requiring that it reproduces the rule base as precise as possible and that it minimizes a parameter depending on its smoothness.

Keywords: fuzzy if-then rule bases, smoothing splines, fuzzy transforms

1. Introduction

A fuzzy if-then rule base typically appears in the following context. A value ranging over a subset Ω of the $\mathbb{R}^m$ is assumed to determine a second value ranging over a subset Ψ of the $\mathbb{R}^n$; we know, however, about this dependency only roughly. Namely, all available information is contained in rules $(\mathbf{A}_1, \mathbf{B}_1), \ldots, (\mathbf{A}_k, \mathbf{B}_k)$, telling us that, for $i = 1, ..., k$, the fuzzy set $\mathbf{A}_i$ over Ω is assigned the fuzzy set $\mathbf{B}_i$ over Ψ.

So a fuzzy if-then rule base may be viewed as a partial function from $\mathcal{F}(\Omega)$, the collection of fuzzy sets over Ω, to $\mathcal{F}(\Psi)$, the collection of fuzzy sets over Ψ. Now, building a fuzzy inference module on the base of a rule base means extending this partial function to a total one; in some reasonable manner, we have to associate with an arbitrary $\mathbf{A} \in \mathcal{F}(\Omega)$ some $\mathbf{B} \in \mathcal{F}(\Psi)$, the cases included that $\mathbf{A}$ is not among the left entries of the rule base.

Most of the existing inference mechanisms are based on the *Generalized Modus Ponens*, realized by the *Compositional Rule of Inference*.[1] These mechanisms differ from each other in the way a fuzzy relation interpreting the fuzzy rule base is constructed, and in the choice of the operations. Two basic examples how a fuzzy relation may be calculated are

$$R = \bigvee_{i=1}^{k} (\mathbf{A}_i \ \mathbf{t} \ \mathbf{B}_i) \quad \text{(DNF)} \quad \text{and} \quad R = \bigwedge_{i=1}^{k} (\mathbf{A}_i \rightarrow_{\mathbf{t}} \mathbf{B}_i) \quad \text{(CNF)},$$

where $\mathbf{t}$ is an arbitrary left-continuous t-norm and $\rightarrow_{\mathbf{t}}$ its residuation.[2] If the minimum t-norm appears in (DNF) we get the Mamdani-Assilian inference algorithm,[3] which is among the algorithms most often implemented in controlling devices.

Another inference technique was proposed by Novak.[4] In this case, the rule base is considered to be a list of independent implicative rules based on the Łukasiewicz implication, and in each inference step, usually only one rule is fired. This method is different from (CNF) above because there the rules are not aggregated by some conjunction. Note that this method has been designed for special fuzzy sets with linguistic content, and it is deeply related to the theory of evaluating linguistic expressions.[5]

A further inference method, which is often used in applications because of its approximation abilities, is due to Takagi-Sugeno.[6] This method assigns a linear function to each antecedent fuzzy set $\mathbf{A}_i$; the result of the inference, however, is already crisp. For this reason, it is actually not directly related to this paper.

On the other hand, there are methods which do come close to our ideas. Note that all inference techniques mentioned so far provide best results when the rule base is dense. If, on the other hand, the rule base is sparse, the KH (Kóczy-Hirota) interpolation methods may be the proper choice, which is able to interpolate between the fuzzy input values.[7,8,9] Other proposals pointing into a similar direction include the one of Jenei.[10]

In this paper, we shall present still another method on which fuzzy inference may be based. Our idea is to disregard for the moment practical considerations and to ask instead for general principles which a function mapping between universes of fuzzy sets and associated to a given rule base should fulfil. We wonder which way we may measure the quality of such a function. What we will do is to define two parameters; the first one will measure how much our function differs from the given rule base, and the second one will be the smaller the smoother the function is. The natural aim is to find the function minimizing these parameter. We will present the conditions under which a unique such function exists; to this end, we will adapt the formalism of smoothing splines to our needs.

Clearly, our approach is different from those mentioned before. The main point is that we intend to demonstrate that it is possible to base a type of fuzzy inference on a certain, reasonably chosen principle. What we do not claim is that our method is superior to other ones. For a short discussion of advantages and disadvantages of our approach, we rather refer to the concluding Section 4.

We proceed as follows. First of all, we will see how we may parametrize a fuzzy set by a finite set of reals, using the concept of fuzzy transforms by I. Perfilieva

(Section 2).[11,12,13,14] Our task is then reduced to find a function between the two parameter spaces, which are finite products of the real unit interval. We then give a short introduction into the method of smoothing splines; a convenient abstract formulation of this formalism is due to Atteia (Section 3).[15] Applying this formalism, we define a function between the parameter spaces which on the one hand is as smooth as possible and which on the other hand reproduces the given if-then rule base as well as possible. These two requirements may be weighted relatively to each other arbitrarily. We conclude the paper with some general considerations (Section 4).

2. Fuzzy transforms

Let us assume that it is our task to design a fuzzy inference module which maps fuzzy sets over a subset Ω of the $\mathbb{R}^m$ to fuzzy sets over a subset Ψ of the $\mathbb{R}^n$. The base sets Ω and Ψ will always be assumed to be compact convex subsets; and a fuzzy set will be understood to be a lower semicontinuous functions from their base set, i.e. Ω or Ψ, to the real unit interval $[0, 1]$. The collections of all fuzzy sets over Ω and Ψ will be denoted by $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Psi)$, respectively.

According to the most general approach, a fuzzy inference module would be a function from $\mathcal{F}(\Omega)$ to $\mathcal{F}(\Psi)$. However, due to the size of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Psi)$, this is clearly a problematic assumption; at least we do not know about a reasonable theoretical framework how to deal with function between such general universes of fuzzy sets. On the other hand, from the practical point it seems inappropriate anyhow to take into account such large collections of functions. So what we shall do is to choose subsets of $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Psi)$ in a reasonable way, and to consider a fuzzy inference module as a function between these subsets.

There are surely many ways how to proceed, and we describe one possibility in this section. Namely, we will make use of a technique proposed by I. Perfielieva.[11,14] By her technique of fuzzy transforms, we may represent each fuzzy set by a finite number of reals; we replace the universes $\mathcal{F}(\Omega)$ and $\mathcal{F}(\Psi)$ by the collection of all the fuzzy transforms. Every fuzzy transform is a r-tuple of reals between 0 and 1, and rather than dealing e.g. with $\mathcal{F}(\Omega)$, we will deal with the parameter set $[0, 1]^r$. So what we will have to consider in the subsequent section are functions from $[0, 1]^r$ to $[0, 1]^s$, where $r, s \in \mathbb{N}$, rather than from $\mathcal{F}(\Omega)$ to $\mathcal{F}(\Psi)$.

Perfilieva's technique may be described as follows. For a given cubic base set Ω, we start with a system of r fuzzy sets called basic functions; for each fuzzy set u, we define a linear combination of the basic functions which is as close as possible to u; the r-tuple of coefficients in this linear combination is then the fuzzy transform. The precision with which we may approximate an arbitrary fuzzy set in this way certainly becomes the better the larger we choose r.

In detail, this works as follows. The basic functions will form a partition of the unity, a notion to be defined first.

Definition 1. Let $k \geq 2$, and let Ω be a closed convex subset of $\mathbb{R}^n$. A *partition of* *the unity over* Ω is a k-tuple $\mathcal{E} = (\mathbf{E}^1, \ldots, \mathbf{E}^k)$ of fuzzy sets over Ω such that the following holds:

(E1) for every i, $\mathbf{E}^i$ is continuous;
(E2) for every i, $\mathbf{E}^i$ attains at some point the value 1;
(E3) $\sum_i \mathbf{E}^i(x) = 1$ for all $x \in \Omega$.

In this case, we call

$$\mathcal{F}(\mathcal{E}) = \{\sum_i A^i \mathbf{E}^i \colon\ 0 \ \leq\ A^1, ..., A^k \ \leq\ 1\}$$

the *fuzzy set universe generated by* $\mathcal{E}$.

We will see next that the fuzzy set universe $\mathcal{F}(\mathcal{E})$ generated by some k-element partition of the unity $\mathcal{E}$, is in a one-to-one correspondence with $[0,1]^k$.

Lemma 1. *Let* $\mathcal{E} = (\mathbf{E}^1, \ldots, \mathbf{E}^k)$ *be a partition of the unity over* $\Omega \subseteq \mathbb{R}^n$. *Then* $\mathcal{F}(\mathcal{E})$, *the fuzzy set universe generated by* $\mathcal{E}$, *consists of fuzzy sets. Moreover, each* $\sum_i A^i \mathbf{E}^i \in \mathcal{F}(\mathcal{E})$ *is uniquely determined by the* k*-tuple* $(A^1, \ldots, A^k)$.

Proof. The first assertion is clear from the fact that the $\mathbf{E}^i$ add up to the identity. Moreover, let $\sum_i A^i \mathbf{E}^i = \sum_i B^i \mathbf{E}^i$ for $A^1, ..., A^k, B^1, ..., B^k \in [0,1]$. For any index i, there is an $x \in \Omega$ such that $\mathbf{E}^i(x) = 1$, whence $\mathbf{E}^j(x) = 0$ for $j \neq i$; it follows $A^i = B^i$. □

Definition 2. Let $k \geq 2$, and let $\mathcal{E}$ be a k-element partition of the unity. Then we call

$$\mathcal{P}(\mathcal{E}) = [0,1]^k$$

the *parameter space belonging to* $\mathcal{E}$. For any

$$\mathbf{A} = A^1 \mathbf{E}^1 + \ldots + A^k \mathbf{E}^k \in \mathcal{P}(\mathcal{E}),$$

we let

$$\bar{\mathbf{A}} = (A^1, \ldots, A^k) \in \mathcal{P}(\mathcal{E}).$$

Now, systems of basic functions are partitions of the unity subject to certain special conditions.

Definition 3. Let $\Omega = [a,b]$ be a closed real interval, let $k \geq 2$, and let $\mathcal{E} = (\mathbf{E}^1, \ldots, \mathbf{E}^k)$ be a partition of the unity over Ω. Let $h = \frac{b-a}{k}$ and $c_0 = a,\ c_1 = a+h, \ldots, c_k = b$. Then $\mathcal{E}$ is called a *system of basic functions over* Ω if the following holds for $i = 1, \ldots, k$:

(E4) $\mathbf{E}^i(c_i) = 1$ and the support of $\mathbf{E}^i$ is within $[c_i - h, c_i + h] \cap \Omega$;
(E5) $\mathbf{E}^i$ is on $[c_i - h, c_i] \cap \Omega$ strictly increasing and on $[c_i, c_i + h] \cap \Omega$ strictly decreasing;

(E6) $\mathbf{E}^i(c_i - h') = \mathbf{E}^i(c_i + h')$ if $0 \leq h' \leq h$ and $c_i - h', c_i + h' \in \Omega$;
(E7) $\mathbf{E}^{i+1}(x) = \mathbf{E}^i(x - h)$ if $1 \leq i < k$ and $x - h, x \in \Omega$.

Furthermore, let $n \geq 1$, let $\Omega_j = [a_j, b_j]$ for every $j = 1, \ldots, n$ be a closed real interval, and let $\Omega = \Omega_1 \times \ldots \times \Omega_n$; we call such a set a *cube*. For every $j = 1, \ldots, n$, let $k_j \geq 2$, and let $(\mathbf{E}_j^1, \ldots, \mathbf{E}_j^{k_j})$ be a system of basic functions. Let

$$\mathbf{E}^{i_1,\ldots,i_n}(x) = \mathbf{E}_1^{i_1}(x) \cdot \ldots \cdot \mathbf{E}_n^{i_n}(x), \quad x \in \Omega,$$

where $1 \leq i_1 \leq k_1, \ldots, 1 \leq i_n \leq k_n$; the $k_1 \cdot \ldots \cdot k_n$-tuple of all these functions (e.g. in the lexicographical order) is again called a *system of basic functions over* Ω.

Let $\mathcal{E}$ a system of basic functions over Ω; with an arbitrary fuzzy $\mathbf{A}$ set over Ω, we may then associate an element of the fuzzy set universe generated by $\mathcal{E}$ as follows.

Definition 4. Let $n \geq 1$, let $\Omega \subseteq \mathbb{R}^n$ be a cube, and let $\mathcal{E} = (\mathbf{E}^1, \ldots, \mathbf{E}^k)$ be a system of basic functions over Ω. Let $\mathbf{A}$ be a fuzzy set over Ω. Then we call the k-tuple of real numbers

$$\mathrm{F}_{\mathcal{E}}[\mathbf{A}] = (A^1, \ldots, A^k)$$

the *F-transform* of $\mathbf{A}$ w.r.t. $\mathcal{E}$, where for $i = 1, \ldots, k$

$$A^i = \frac{\int_\Omega \mathbf{A}(x)\mathbf{E}^i(x)dx}{\int_\Omega \mathbf{E}^i(x)dx}. \tag{1}$$

Moreover, call

$$\mathrm{IF}_{\mathcal{E}}[\mathbf{A}](x) = \sum_{i=1}^{k} A^i \mathbf{E}^i(x)$$

the *inverse F-transform* associated to $\mathbf{A}$ w.r.t. $\mathcal{E}$.

Now, an arbitrary fuzzy set $\mathbf{A}$ is approximated by its fuzzy transform $\mathrm{F}_{\mathcal{E}}[\mathbf{A}]$; this is seen from the following theorem. The point is simply that a continuous fuzzy set $\mathbf{A}$ differs from its inverse F-transform $\mathrm{IF}_{\mathcal{E}}[\mathbf{A}]$ w.r.t. the supremum norm by an arbitrary small positive real if the cardinality of $\mathcal{E}$ is sufficiently large. Recall furthermore that an arbitrary fuzzy set can in turn be approximated arbitrarily close by a continuous fuzzy set for instance w.r.t. the L^2-norm.

For further details, we refer to the cited papers.[11,14]

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^n$ be a cube, and let $\mathbf{A}$ be a continuous fuzzy set over Ω. Then for any $\varepsilon > 0$ there exists a system of basic functions $\mathcal{E}$ such that*

$$|\mathbf{A}(x) - IF_{\mathcal{E}}[\mathbf{A}](x)| \leq \varepsilon \text{ for all } x \in \Omega.$$

Proof. This was proven earlier.[16,12,14] □

As a consequence, rather than working with the large universe $\mathcal{F}(\Omega)$ of fuzzy sets of some cube Ω, we may restrict to the fuzzy set universe generated by some fixed system of basic functions over Ω.

3. Smoothing splines for fuzzy if-then rule bases

After the preparations of Section 2, let us recall our original aim. Assume that we are given a fuzzy if-then rule base $(\mathbf{A}_1, \mathbf{B}_1), \ldots, (\mathbf{A}_l, \mathbf{B}_l)$, where $\mathbf{A}_1, ..., \mathbf{A}_l$ are fuzzy sets over $\Omega \subseteq \mathbb{R}^m$ and $\mathbf{B}_1, ..., \mathbf{B}_l$ are fuzzy sets over $\Psi \subseteq \mathbb{R}^n$; and assume that we have to determine a function from the fuzzy sets over Ω to the fuzzy sets over Ψ in accordance with this rule base.

Since Ω and Ψ were assumed to be bounded, we may actually assume that they are cubes. Let us moreover fix a system of basic functions $\mathcal{E}_A$ over Ω, and a system of basic functions $\mathcal{E}_B$ over Ψ. By choosing the cardinality of $\mathcal{E}_A$ and $\mathcal{E}_B$ as large as necessary, it is the content of Theorem 1 that it is no serious restriction to assume that $\mathbf{A}_1, ..., \mathbf{A}_l$ are in the fuzzy set universe generated by $\mathcal{E}_A$ and that $\mathbf{B}_1, ..., \mathbf{B}_l$ are in the fuzzy set universe generated by $\mathcal{E}_B$ (cf. Definition 1). So we may identify $\mathbf{A}_i$ with $\bar{\mathbf{A}}_i \in [0,1]^r$ and similarly $\mathbf{B}_i$ with $\bar{\mathbf{B}}_i \in [0,1]^s$, $i = 1, \ldots, l$ (Lemma 1 and Definition 2).

As a result, our task is reduced to finding an appropriate function between the respective parameter spaces $\mathcal{P}_A = \mathcal{P}(\mathcal{E}_A) = [0,1]^r$ to $\mathcal{P}_B = \mathcal{P}(\mathcal{E}_B) = [0,1]^s$. Namely, we have to find a function $f\colon \mathcal{P}_A \to \mathcal{P}_B$ subject to the following conditions:

- f should be monotonous, that is, f should preserve the usual partial order of fuzzy sets. So if $\bar{\mathbf{A}}, \bar{\mathbf{A}}' \in \mathcal{P}_A$, then $\bar{\mathbf{A}} \leq \bar{\mathbf{A}}'$ should imply $f(\bar{\mathbf{A}}) \leq f(\bar{\mathbf{A}}')$. Here, $\leq$ denotes the componentwise ordering of $[0,1]^r$ and $[0,1]^s$, respectively.
- The function f should be "as smooth as possible". We interprete this condition in the usual way: For some q, we assume that the q-th (generalized) derivative $D^q f$ exists and is L^2-integrable; and the L^2-norm of $D^q f$ is to be minimized.
- For $i = 1, ..., l$, $f(\bar{\mathbf{A}}_i)$ should be as close as possible to $\bar{\mathbf{B}}_i$. This means that the distances between $f(\bar{\mathbf{A}}_i)$ and $\bar{\mathbf{B}}_i$ are to be minimized as well.

The two latter requirements may be arbitrarily weighted relatively to each other; the controlling constant is ρ, appearing below. We note that the last requirement may in principle be strengthened; we could require $f(\bar{\mathbf{A}}_i) = \bar{\mathbf{B}}_i$ for all i. This would require a slightly more involved procedure.[17]

Our framework is the following. First of all, F, formally defined below, denotes the Hilbert space containing those functions which map one parameter space to the other one and which are of the appropriate differentiability class. Now, in our case, these functions need to have the monotonicity property; consequently, we will restrict to a certain positive cone in F, denoted by K. Second, the differential operator δ will map F into another Hilbert space D, such that for $f \in F$ the norm of $\delta(f)$ in D will measure the degree of smoothness of f. Third, there will be another linear function ω from F to a Hilbert space V which maps $f \in F$ to the values of f at those points whose images are known from the rule base.

In what follows, we will distinguish the scalar products and the 0's of the three

Hilbert spaces by indices referring to the respective space.

The following theorem provides a variant of the well-known smoothing spline formalism; for a more detailed version of its proof, see Chapter III of Atteia's monograph.[15]

Theorem 2. *Let F, D, and V be Hilbert spaces; let $K \subseteq F$ be convex and closed; let $\delta\colon\ F \to D$ and $\omega\colon\ F \to V$ be linear operators such that δ has a closed image and ω is surjective; let $v_0 \in V$; and let $\rho > 0$. Assume furthermore that $\omega(Ker\ \delta)$ is closed in V and that $Ker\ \delta \cap Ker\ \omega = \{0_F\}$. Then there is unique $f \in K$ which minimizes*

$$||\delta(f)||_D + \rho||\omega(f) - v_0||_V, \quad \text{where } f \in K. \tag{2}$$

Proof. Let $D' = \delta(F)$; then D' is a closed subspace of D. Let $H = D' \times V$ the Hilbert space with the scalar product

$$((d_1, v_1), (d_2, v_2))_H \ = \ (d_1, d_2)_D + \rho(v_1, v_2)_V,$$

for $d_1, d_2 \in D'$ and $v_1, v_2 \in V$, and let

$$\ell\colon\ F \to H, \quad f \mapsto (\delta(f), \omega(f)).$$

Then by III, Lemma 2.1 in Atteia's monograph,[15] ℓ is a isomorphism between F and the closed subspace $\ell(F)$. Consequently, the restriction $\ell|_K$ of ℓ to K is a homeomorphism between K and $\ell(K)$ preserving convex combinations in both directions. In particular, $\ell(K)$ is a closed convex subset of H. Let $(d, v) \in \ell(K)$ be the unique projection of $(0, v_0)$ onto $\ell(K)$. Then $f = \ell^{-1}((d, v))$ minimizes (2). □

We now apply Theorem 2 to functions mapping between the two parameter spaces $\mathcal{P}_A$ and $\mathcal{P}_B$. Moreover, let $q \geq 1$ be some integer, and let $\overset{\circ}{\mathcal{P}}_A$ be the interior of $\mathcal{P}_A$; then we denote by $W^{q,2}(\overset{\circ}{\mathcal{P}}_A, \mathbb{R}^s)$ the Sobolev space of functions from $\overset{\circ}{\mathcal{P}}_A$ to $\mathbb{R}^s$, endowed with the norm $|| \cdot ||_{q,2}$.[18] Note that since $\overset{\circ}{\mathcal{P}}_A$ fulfils the strong local Lipschitz condition, then this space consists of continuous functions continuously extendible to $\mathcal{P}(\mathcal{E}_A)$ if $q \geq \frac{r+3}{2}$.[18]

Elements $a_1, \ldots, a_l \in \mathbb{R}^r$ are called q-unisolvent if every polynomial of degree $\leq q - 1$ vanishing at $a_1, \ldots, a_l$ is 0.

Theorem 3. *Let $\mathcal{E}_A$ be an r-element system of basic functions over a cube $\Omega \subseteq \mathbb{R}^m$, and let $\mathcal{E}_B$ be an s-element system of basic functions over a cube $\Psi \subseteq \mathbb{R}^n$. Let $(\mathbf{A}_1, \mathbf{B}_1), \ldots, (\mathbf{A}_l, \mathbf{B}_l) \in \mathcal{F}(\mathcal{E}_A) \times \mathcal{F}(\mathcal{E}_B)$.*

Let $\mathcal{P}_A = \mathcal{P}(\mathcal{E}_A) = [0, 1]^r$ and $\mathcal{P}_B = \mathcal{P}(\mathcal{E}_B) = [0, 1]^s$.

Let $q \geq \frac{r+3}{2}$ and

$$\begin{aligned} F &= W^{q,2}(\overset{\circ}{\mathcal{P}}_A, \mathbb{R}^s) \\ &= \{f \in L^2(\overset{\circ}{\mathcal{P}}_A, \mathbb{R}^s)\colon\ D^t f \in L^2(\overset{\circ}{\mathcal{P}}_A, S((\mathbb{R}^r)^t, \mathbb{R}^s) \text{ for all } t \leq q\}, \\ K &= \{f \in F\colon\ f \text{ is monotonous and } f(\overset{\circ}{\mathcal{P}}_A) \subseteq \mathcal{P}_B\}, \end{aligned}$$

where $S((\mathbb{R}^r)^t, \mathbb{R}^s)$ *is the space of symmetric t-linear forms from* $\mathbb{R}^r$ *to* $\mathbb{R}^s$*. Furthermore, let*

$$D = L^2(\overset{\circ}{\mathcal{P}}_A, S((\mathbb{R}^r)^q, \mathbb{R}^s)),$$
$$\delta\colon\ F \to D,\quad f \mapsto D^q f;$$
$$V = (\mathbb{R}^s)^l,$$
$$\omega\colon\ F \to V,\quad f \mapsto (\hat{f}(\bar{\mathbf{A}}_1), \ldots, \hat{f}(\bar{\mathbf{A}}_l)),$$

where $\hat{f}$ *is the continuous extension of* f *to* $\mathcal{P}_A$*.*

Assume that $\bar{\mathbf{A}}_1, \ldots, \bar{\mathbf{A}}_l$ *is q-unisolvent, and let* $v_0 = (\bar{\mathbf{B}}_1, \ldots, \bar{\mathbf{B}}_l)$*. Then, for any* $\rho > 0$*, there is a unique* $f \in K$ *minimizing (2).*

Proof. We claim that δ has a closed image. Indeed, $f \mapsto ||\delta(f)||_D$ is a seminorm on F, which induces a norm on $F/\mathcal{P}_q$, where $\mathcal{P}_q$ is the subspace of polynomials of degree $\leq q-1$. The assertion follows from the fact that $F/\mathcal{P}_q$ is a Hilbert space.

Moreover, ω is obviously surjective. Ker $\delta = \mathcal{P}_q$; hence Ker $\delta \cap$ Ker $\omega = \{0_F\}$. It furthermore follows that ω(Ker δ) is closed. K is obviously convex and closed. So we may apply Theorem 2. □

This result may be put into simple words as follows. Given two parameter spaces $\mathcal{P}_A$ and $\mathcal{P}_B$, describing up to some prescribed precision the inputs and outputs, respectively, of a fuzzy inference module, and given a fuzzy if-then rule base with entries in the corresponding fuzzy set universes, there is a unique function $\hat{f}\colon\ \mathcal{P}_A \to \mathcal{P}_B$ minimizing the weighted sum of the smoothness degree and the error in reproducing the rule base – under the condition that the rule base has sufficiently many entries.

4. Discussion

We described in this paper a method how to realize a fuzzy inference module based on an arbitrary fuzzy if-then rule base.

As the main feature of this method, we should exhibit the fact that it provides the completion of a rule base according to a clear, general principle. In particular, it is applicable nearly unrestrictedly; there is no serious restriction of the shape of the fuzzy sets involved, nor is there any restriction in the dimensions.

On the other side, generality certainly has its drawback, and in comparison to all methods mentioned in the introduction, certain disadvantages have to be admitted. To implement our method is as difficult as to implement the method of smoothing splines in general: it is surely technically demanding. Moreover, there is a serious point restricting the applicability of our method: there must be sufficiently many rules – approximately as many as one half of the number of basic functions used to represent the input fuzzy sets.

We conclude the paper by having a look to the closely related work of the first author.[17] The approach of this work is very similar, but in the cited paper, the fuzzy sets are parametrized differently: finitely many level sets are chosen and the support functions of these level sets are approximated by their values at finitely many points of the unit sphere. The advantage is that these parameters are compatible with the arithmetic operations of fuzzy sets, that is, with the addition and with the multiplication with positive reals. As a consequence, when the values in a rule base become sharper and sharper, the resulting function will in some sense approach the function which you would get in the crisp case. On the other side, it is a disadvantage in the other paper[17] that the parameter set, which has the simple form $[0,1]^r$ here, is not described so easily. Finally, only fuzzy-convex fuzzy sets (in the sense of Diamond and Kloeden[19]) are taken into acc! ount.

Acknowledgements. Both authors' work was supported by the German Research Foundation (DFG) as part of the Collaborative Research Center "Computational Intelligence" (SFB 531, subproject A1). Furthermore, M.Š. acknowledges the partial support by the grant IAA 1187301 GAAV ČR and by the project MSM6198898701 of the MŠMT ČR.

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