Pseudo-BCK algebras as partial algebras

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September 2009

Abstract

It is well-known that the representation of several classes of residuated lattices involves lattice-ordered groups. An often applicable method to determine the representing group (or groups) from a residuated lattice is based on partial algebras: the monoidal operation is restricted to those pairs which fulfil a certain extremality condition, and else left undefined. The subsequent construction applied to the partial algebra is easy, transparent, and leads directly to the structure needed for representation.

In this paper, we consider subreducts of residuated lattices, the monoidal and the meet operation being dropped: the resulting algebras are pseudo-BCK semilattices. Assuming divisibility, we can pass on to partial algebras also in this case. To reconstruct the underlying group structure from this partial algebra, if applicable, is again straightforward. We demonstrate the elegancy of this method for two classes of pseudo-BCK semilattices: semilinear divisible pseudo-BCK algebras and cone algebras.

Keywords: Pseudo-BCK algebras, non-commutative fuzzy logics, partially ordered groups.

1 Introduction

In the area of fuzzy logics, the interest in residuated lattices has considerably increased during the last years. Indeed, both fields are closely related; the algebraic semantics of numerous fuzzy logics is based on a subvariety of the variety of residuated lattices. The latter reflect the basic properties fulfilled by a set of propositions considered in the framework of a fuzzy logic: the lattice order models the relative strength of propositions, the monoidal operation models the conjunction of two propositions, and the implication represents the minimal additional information needed to infer a proposition from another one.

In this paper, we are concerned with a special technique to represent the implicational subreducts of residuated lattices. The technique rests upon the close relationship between residuated lattices and ℓ- (lattice-ordered) groups.

Recall that for several classes of the residuated lattices, theorems exist according to which the algebras arise from an ℓ-group, or from a collection of ℓ-groups, in a specific way. For instance, a convex subset of an ℓ-group, like the negative cone or an interval of it, can serve as the base set and the monoidal operation be defined by means of the group operation. Algebras constructed in this way can be further combined by means of direct products or ordinal sums or combinations of both.

The best known examples of residuated lattices where ℓ-groups are used for representation, are probably MV-algebras. Recall that the variety of MV-algebras is the algebraic counterpart of the Łukasiewicz infinite-valued logic. Let $(G; \wedge, \vee, \cdot, 1)$ be a (multiplicatively written) Abelian ℓ-group, and let u be an element of its negative cone. Let $L = \{g \in G \colon\ u \leq g \leq 1\}$, and define, for any $a, b \in L$,

$$a \circ b \;=\; a \cdot b \;\vee\; u,$$
$$b \rightarrow a \;=\; a \cdot b^{-1} \;\wedge\; 1.$$

Then $(L; \wedge, \vee, \circ, \rightarrow, u, 1)$ is an MV-algebra, and any MV-algebra arises in this way [33]. Further examples where ℓ-groups are used include pseudo-MV algebras [7, 12], BL-algebras and hoops [5, 1], as well as pseudo-BL algebras and pseudohoops [8].

It is not a widely known fact that the group representation is in all these cases particularly easy to derive by use of partial algebras. To get an idea what we mean, consider the MV-algebra $([\frac{1}{2}, 1]; \wedge, \vee, \circ, \rightarrow, 1, 0)$ where $[\frac{1}{2}, 1] = \{r \in \mathbb{R} : \frac{1}{2} \leq r \leq 1\}$ is endowed with the natural lattice order, the truncated product $\circ \colon\ [\frac{1}{2}, 1]^2 \rightarrow [\frac{1}{2}, 1],\ \ (a, b) \mapsto ab \vee \frac{1}{2}$, the truncated quotient $\rightarrow \colon\ [\frac{1}{2}, 1]^2 \rightarrow [\frac{1}{2}, 1],\ \ (a, b) \mapsto \frac{b}{a} \wedge 1$, and the constants 0 and 1. Clearly, this MV-algebra arises from $(\mathbb{R}^+ \backslash \{0\}; \wedge, \vee, \cdot, 1)$, the multiplicative group of strictly positive reals, in the way explained above. Note now that the monoidal operation $\circ$ coincides with the group multiplication whenever the group product does not fall below the bottom element. Equivalently, we may state that $a \circ b = ab$ for $a, b \in [\frac{1}{2}, 1]$ if and only if the following condition is fulfilled: a is the largest element x such that $x \circ b = a \circ b$, and b is the largest element y such that $a \circ y = a \circ b$. In other words, we have $a \circ b = ab$ exactly if whenever a is replaced by a larger element, the product will be larger as well, and similarly for b.

Accordingly, we associate to this MV-algebra the partial algebra $([\frac{1}{2}, 1]; \vee, \cdot, 1)$, where the meet operation and the constant 1 is taken from the original algebra, but the monoidal operation is replaced by the partial binary operation $\cdot$, which is the restriction of $\circ$ to the pairs fulfilling the mentioned maximality condition. So $a \cdot b$ is defined as the usual product of reals if and only if this product is not smaller than $\frac{1}{2}$, and else undefined.

Up to order-theoretical duality, $([\frac{1}{2}, 1]; \vee, \cdot, 1)$ is a lattice-ordered effect algebra [15], whose crucial property is cancellativity: if $a \cdot b$ and $a \cdot c$ are defined and coincide, it follows $b = c$. Let now $(G; \cdot, 1)$ be the Abelian group freely generated by $[\frac{1}{2}, 1]$ subject to the condition that $a \cdot b = c$ if this equation holds in the partial algebra. It is not difficult to see that G is isomorphic to the multiplicative group of strictly positive reals. In particular, the natural embedding of $[\frac{1}{2}, 1]$ into G is injective, and G can be linearly ordered in a way that $[\frac{1}{2}, 1]$ generates $G^- = (0, 1]$ as a semigroup. It follows that, as we will say, $([\frac{1}{2}, 1]; \vee, \cdot, 1)$ is isomorphically embeddable into the ℓ-group $(\mathbb{R}^+ \backslash \{0\}; \wedge, \vee, \cdot, 1)$.

The problem to embed a partial groupoid into a group has probably the first time systematically studied by Baer [2]. Effect algebras are special partial groupoids; they are strongly associative, commutative, cancellative, naturally ordered, and bounded w.r.t. this order. For effect algebras, Baer's method was elaborated in [35] and is applicable to any MV-algebra. But MV-algebras are only one example where the method works; the procedure can be generalized. For instance, the presence of a bottom element is not essential; the monoidal operation need not to be assumed to be commutative; the assumption that the partial order is a lattice order can be dropped. As an essential condition, an analogon of the Riesz decomposition property of partially ordered groups remains. These facts have been exhibited e.g. in [12, 13, 14].

A partial algebra can actually be associated in the way indicated above to *any* integral residuated lattice, with the effect that the total algebra is uniquely determined by the partial algebra.

The full significance of this observation has not yet been explored. It has been applied, e.g., to BL-algebras, and the result is a short and transparent new proof of the representation theorem for these algebras [37]: the total algebras are used only to show that BL-algebras are semilinear; the partial algebras associated to linearly ordered BL-algebra are easily seen to be ordinally composed from linearly ordered generalized effect algebras; and the latter embed into linearly ordered Abelian groups.

In this paper, we deal with pseudo-BCK semilattices. These are pseudo-BCK algebras whose partial order is a join-semilattice. Such an algebra, say $(L; \vee, \backslash, /, 1)$, arises from a residuated lattice by dropping the monoidal and the meet operation and by possibly restricting to a subalgebra; this fact has been established by J. Kühr [28]. The method of using partial algebras to represent pseudo-BCK semilattices, however, is directly applicable, that is, without the need to embed into a residuated lattice; cf. [32, Section 4.3]. To be able to define the partial algebras, all what we have to assume is divisibility, a property defined analogously to residuated lattices.

It is open if a complete analysis of pseudo-BCK semilattices on the basis of po- (partially ordered) groups is possible under the assumption of divisibility alone. A promising idea to study this problem is the poset sum construction, which was introduced in [26]. Here, we shall present the line of argumentation needed to represent (i) linearly ordered divisible pseudo-BCK algebras, and (ii) cone algebras. In case (i), we give an alternative proof of the representation theorem of A. Dvurečenskij and J. Kühr [10]; the case (ii) leads to an alternative and in fact significantly optimized proof of B. Bosbach's result [4] that cone algebras embed into ℓ-groups.

The paper is organized as follows. After recalling some basic facts about pseudo-BCK semilattices (Section 3), we define the partial algebras associated to them, and we compile a list of properties shared by these partial algebras (Section 2). We then recall the representation theory of a specific type of partial algebras, known under the name generalized pseudoeffect algebras (Section 4). Subsequently, we apply the method. An easy case are cone algebras (Section 5). Second, we turn to linearly ordered divisible pseudo-BCK algebras, which are the implicational counterparts of pseudohoops, which in turn include pseudo-BL algebras (Section 6). In the last part (Section 7), we add the corollary that the considered pseudo-BCK algebras embed into residuated lattices of the corresponding kind.

A word is in order concerning the notation used in this article. There are two competing ways to define a residuated lattice: based on the residual triple $\circ, \backslash, /$ where $a \backslash b$ is the maximal element x such that $a \circ x \leq b$ and similarly for b/a; or based on the triple $\oplus, \oslash, \oslash$, where $b \oslash a$ is the minimal element x such that $a \oplus x \geq b$ and similarly for $b \oslash a$. In the latter case, we are led to representations, if applicable, such that $\oplus$ corresponds to the group addition and $\oslash, \oslash$ to the left and right difference, respectively. Consider the dual of the example of an MV-algebra above: the algebra $([0,1]; \wedge, \vee, \oplus, \ominus, 0)$ where $[0,1]$ is the real unit interval endowed with the natural order and the operations $\oplus \colon\ [0,1]^2 \to [0,1],\ \ (a,b) \mapsto (a+b) \wedge 1$ and $\ominus \colon\ [0,1]^2 \to [0,1],\ \ (a,b) \mapsto (a-b) \vee 0$. The associated partial algebra is $([0,1]; +, 0)$ where $a+b$ is the usual sum of the reals a, b if below 1 and undefined otherwise. The representing group are the reals with addition. Although this picture might look appealing, both in the field of residuated lattices and in fuzzy logics, the dual notions, which we have already used in the illustrating example above, are common. In this article, we adopt the latter choice, and we will do so consistently in the whole article, the case of partial algebras included.

2 Pseudo-BCK semilattices

In this paper, we examine a certain class of pseudo-BCK algebras [23]. As the adjunct “pseudo” suggests, we deal with a non-commutative generalization of BCK-algebras, which in turn are associated to the so-called BCK logic. The BCK logic is distinguished by the fact that it is based solely on the implication connective. For a comprehensive overview of results on pseudo-BCK algebras, we recommend J. Kühr’s Habilitation Thesis [32].

We will make two restrictions, the first one concerning the partial order. Indeed, any pseudo-BCK algebra can be partially ordered in a natural way. Here, we will generally assume that all (finite) suprema exist; we will actually add the supremum as an own operation. Actually, a large part of our considerations would work without this assumption, which however is fulfilled in the interesting cases and has turned out to be convenient in some technical respects.

Definition 2.1. A *pseudo-BCK semilattice* is an algebra $(L; \vee, /, \backslash, 1)$ of type $\langle 2, 2, 2, 0 \rangle$ such that for any $a, b, c \in L$:

(B1) $(L; \vee, 1)$ is a upper-bounded join-semilattice;

(B2) the mappings $x \mapsto x/a$ and $x \mapsto a \backslash x$ are isotone;

(B3) $b \leq a \backslash c$ if and only if $a \leq c/b$;

(B4) $1 \backslash a \;=\; a/1 \;=\; a$;

(B5) $(b \backslash a)/c \;=\; b \backslash (a/c)$.

What we call a pseudo-BCK semilattice should actually be called a “pseudo-BCK join-semilattice”; cf. [31]. We have chosen the shorter notion for convenience.

Unlike the larger class of pseudo-BCK algebras, the class of pseudo-BCK semilattices is a variety. An axiomatization by equations can be found in [32, Section 1.1].

To be able to associate a partial algebra to a pseudo-BCK semilattice, we need a second condition: divisibility.

Definition 2.2. We will call a pseudo-BCK semilattice *divisible* if, for any $a, b, c \in L$:

(D) $(a\backslash b)\backslash(a\backslash c) \;=\; (b\backslash a)\backslash(b\backslash c) \;=\; a\backslash((b/a)\backslash c)$,
$(c/a)/(b/a) \;=\; (c/b)/(a/b) \;=\; (c/(a\backslash b))/a$.

These equations look certainly weird. The motivation becomes clear only when comparing them with the corresponding notion for residuated lattices, where divisibility means $a \wedge b \;=\; a{\circ}(a\backslash b) \;=\; (b/a){\circ}a$, $\circ$ being the monoidal operation (cf. Section 7 below). Taking into account that furthermore $(a{\circ}b)\backslash c \;=\; b\backslash(a\backslash c)$ and similarly $c/(a{\circ}b) \;=\; (c/b)/a$, the choice of the equations (D) becomes plausible.

Lemma 2.3. *The algebra* $(L; \vee, /, \backslash, 1)$ *is a divisible pseudo-BCK semilattice if and only if the axioms* (B1), (B3), (B4), *and equations* (D) *hold.*

We list some basic properties of the algebras under consideration, skipping in some cases the versions with $\backslash$ and $/$ being interchanged.

Lemma 2.4. *Let* $(L; \vee, /, \backslash, 1)$ *be a pseudo-BCK semilattice. Then we have for all* $a, b, c \in L$*:*

(i) $a \leq b$ *if and only if* $a\backslash b = 1$ *if and only if* $b/a = 1$.

(ii) *The mappings* $x \mapsto a/x$ *and* $x \mapsto x\backslash a$ *are antitone.*

(iii) $b \leq (a/b)\backslash a$,

(iv) $c\backslash b \leq (c\backslash a)/(b\backslash a)$,

(v) $b\backslash a \leq (c\backslash b)\backslash(c\backslash a)$,

(vi) $(b/(a\backslash b))\backslash b \;=\; a\backslash b$,

If L *is divisible, we furthermore have:*

(vii) *Let* $a, b \leq c$*. If* $c\backslash b \leq c\backslash a$ *or* $b/c \leq a/c$*, then* $b \leq a$*.*
In particular, if $c\backslash a = c\backslash b$ *or* $a/c = b/c$*, then* $a = b$*.*

(viii) *Let* $a \leq b \leq c$*. Then* $[a/(b\backslash a)]/(c\backslash b) = a/(c\backslash a)$*.*

Proof. We only show (viii). Let $a \leq b \leq c$. By divisibility, $b\backslash a = (c\backslash b)\backslash(c\backslash a)$; so $[a/(b\backslash a)]/(c\backslash b) = [a/((c\backslash b)\backslash(c\backslash a))]/(c\backslash b) = a/(c\backslash a)$ again by divisibility. ☐

3 R-algebras

With any divisible pseudo-BCK semilattice, we may associate a partial algebra as follows.

Definition 3.1. Let $(L; \vee, /, \backslash, 1)$ be a divisible pseudo-BCK semilattice. For $a, b \in L$, we define $a \cdot b = c$ if c is the unique element such that

$$a = c/b \quad \text{and} \quad b = a\backslash c; \tag{1}$$

otherwise, we leave $a \cdot b$ undefined. Then $(L; \vee, \cdot, 1)$ is called the *partial algebra associated to* L. We furthermore define, for $a, b \in L$,

$$\begin{aligned} a \preccurlyeq_r b &\quad \text{if there is an } x \in L \text{ such that } b \cdot x \text{ exists and equals } a, \\ a \preccurlyeq_l b &\quad \text{if there is an } y \in L \text{ such that } y \cdot b \text{ exists and equals } a. \end{aligned} \tag{2}$$

The definition of the partial product makes sense because, for elements a and b, there is at most one element fulfilling the condition (1); this is the content of the subsequent lemma. Moreover, as to be expected, $\preccurlyeq_l$ and $\preccurlyeq_r$ are partial orders; this will be shown only at the end of the present section.

We will adopt the usual convention that statements involving partial operations are meant to comprise the statement that these operations exist. In particular, we say "$a \cdot b = c$" when we mean "$a \cdot b$ is defined and equals c".

Lemma 3.2. *Let L be a divisible pseudo-BCK semilattice, and let $a, b \in L$. If some $c \in L$ fulfils* (1)*, then c is the only element fulfilling* (1)*.*

Proof. Let c be such that (1) holds. Then $c \leq a, b$. So the uniqueness follows by Lemma 2.4(vii). □

We will next demonstrate that the transition from a pseudo-BCK semilattice to its associated partial algebra means no loss of information.

Lemma 3.3. *Let L be a divisible pseudo-BCK semilattice, and let $a, b \in L$.*

(i) $(a/(b\backslash a)) \cdot (b\backslash a) = (a/b) \cdot ((a/b)\backslash a) = a$.

(ii) *There is a smallest element $\bar{b} \geq b$ such that $a \preccurlyeq_r \bar{b}$, namely, $\bar{b} = a/(b\backslash a)$. Similarly, there is a smallest element $\bar{\bar{b}} \geq b$ such that $a \preccurlyeq_l \bar{\bar{b}}$, namely, $\bar{\bar{b}} = (a/b)\backslash a$.*

(iii) $a \preccurlyeq_r b$ *if and only if* $a/(b\backslash a) = b$*. Similarly,* $a \preccurlyeq_l b$ *if and only if* $(a/b)\backslash a = b$*.*

Proof. (i) In view of Lemma 2.4(vi), this is clear from the definition of the partial product.

(ii) We show the first half. Put $\bar{b} = a/(b\backslash a)$; then $\bar{b} \geq b$ and $\bar{b} \cdot (b\backslash a) = a$ by part (i).

Let $b' \geq b$ such that $a \preccurlyeq_r b'$. Then $b' \cdot y = a$ for some y, and $b' = a/(b'\backslash a) \geq a/(b\backslash a) = \bar{b}$ by Lemma 2.4(ii).

(iii) This is obvious by the definition of $\cdot$ and part (i). □

Theorem 3.4. *Let* $(L; \vee, /, \backslash, 1)$ *be a divisible pseudo-BCK semilattice, and let* $(L; \vee, \cdot, 1)$ *be the associated partial algebra. Then the latter structure determines the former uniquely. Namely, for* $a, b \in L$*, let* $\bar{b} \geq b$ *be the smallest element such that* $a \preccurlyeq_r \bar{b}$*; then the unique element* x *such that* $\bar{b} \cdot x = a$ *equals* $b\backslash a$*. Similarly for* a/b*.*

Proof. By Lemma 3.3, $\bar{b} = a/(b\backslash a)$. But from $\bar{b} \cdot x = a$, it follows $x = \bar{b}\backslash a = (a/(b\backslash a))\backslash a = b\backslash a$. □

In other words, an analysis of the partial algebra associated to a divisible pseudo-BCK semilattice means an analysis of the latter. Accordingly, our next aim is to characterize this partial algebra as far as possible.

Definition 3.5. An *R-algebra* is a partial algebra $(L; \vee, \cdot, 1)$ of type $\langle 2, 2, 0 \rangle$ such that for any $a, b, c \in L$:

(E1) $(L; \vee, 1)$ is a upper-bounded join-semilattice.

(E2) $(a \cdot b) \cdot c$ is defined iff $a \cdot (b \cdot c)$ is defined, and in this case $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(E3) $a \cdot 1$ and $1 \cdot a$ is defined and equals a.

(E4) If $a \cdot c$ and $b \cdot c$ are defined, then $a \leq b$ if and only if $a \cdot c \leq b \cdot c$.

If $c \cdot a$ and $c \cdot b$ are defined, then $a \leq b$ if and only if $c \cdot a \leq c \cdot b$.

(E5) Let $a \leq b$. Then there is a smallest element $\bar{b} \geq b$ such that $\bar{b} \cdot x = a$ for some $x \in L$.

Similarly, there is a smallest element $\bar{\bar{b}} \geq b$ such that $y \cdot \bar{\bar{b}} = a$ for some $y \in L$.

(E6) If $a \cdot b \leq c \leq a$, there is an $x \in L$ such that $c = a \cdot x$.

Similarly, if $a \cdot b \leq c \leq b$, there is a $y \in L$ such that $c = y \cdot b$.

On an R-algebra, we define the relations $\preccurlyeq_r$ and $\preccurlyeq_l$ according to (2).

Furthermore, an R-algebra will be called *normal* if for any $a, b \in L$:

(N) If $a \cdot b$ is defined, there are $x, y \in L$ such that $a \cdot b = x \cdot a = b \cdot y$.

As a reader with background in quantum structures will notice, we have broken with notational and terminological conventions. For partial algebras related to those discussed here, commonly the dual order is taken and accordingly the partial operation written as $+$. Moreover, in the dual notation, an algebra fulfilling (E1)–(E5) and (N) would be called a "lower-semilattice-ordered weak generalized pseudoeffect algebra". This expression is cumbersome, the background being the unfortunate practice to specify a partial algebra not by what holds, but by what is not assumed, the structure of reference being effect algebras. Namely, when comparing R-algebras to effect algebras [15], we see that R-algebras are not necessarily commutative ("pseudo"), that there is not necessarily a bottom element ("generalized"), and that the partial order $\leq$ is not the one induced by the product in the natural way ("weak"). We note furthermore that axiom (E6) has not been used as a basic axiom before (cf. [37]).

Theorem 3.6. *Let $(L; \vee, /, \backslash, 1)$ be a divisible pseudo-BCK semilattice. Then the associated algebra $(L; \vee, \cdot, 1)$ is an R-algebra.*

Proof. (E1) holds by (B1).

(E3) holds by (B4) and Lemma 2.4(i).

To see (E4), let $a \cdot c$ and $b \cdot c$ be defined. If $a \leq b$, then $(a \cdot c)/c = a \leq b = (b \cdot c)/c$, so $a \cdot c \leq b \cdot c$ by Lemma 2.4(vii). Conversely, if $a \cdot c \leq b \cdot c$, we conclude $a \leq b$ by (B2). So the first part of (E4) follows, and the second part is proved similarly.

(E5) holds by Lemma 3.3(iii).

To see (E6), assume $a \cdot b \leq c \leq a$. Put $d = a \cdot b$. By divisibility, $c \backslash d = (a \backslash c) \backslash (a \backslash d)$, and by Lemma 2.4(viii), it follows $a \leq c/(a \backslash c) \leq (d/(c \backslash d))/(a \backslash c) = d/(a \backslash d) = a$. So $a = c/(a \backslash c)$, whence by Lemma 3.3(iii) $a \cdot (a \backslash c) = c$. By (E4), $b \leq a \backslash c$. This completes one half of (E6); the other one is shown similarly.

We finally show (E2). Assume that $(a \cdot b) \cdot c$ is defined. Put $e = a \cdot b$ and $d = (a \cdot b) \cdot c$. Let $f = a \backslash d$. Then $e \cdot c \leq f \leq c$ by Lemma 2.4(ii), so by (E6) $f = b' \cdot c$ for some $b' \geq e$. We have $b' = f/c = (a \backslash d)/c = a \backslash (d/c) = b$; so we have shown that

$f = a\backslash d = b \cdot c$. Now, let $a' = d/f$. Then $d = a' \cdot f$ because $a'\backslash d = (d/f)\backslash d = a\backslash d = f$. We have $a = (d/c)/b = (d/c)/(f/c) \geq d/f = a' \geq a$; so the proof is complete that $a \cdot (b \cdot c)$ is defined and equals d. Under the assumption that $a \cdot (b \cdot c)$ is defined, we proceed similarly, and the associativity axiom (E2) follows. □

In view of this theorem, we will w.r.t. a divisible pseudo-BCK semilattice refer in the sequel to the "associated R-algebra" rather than the "associated partial algebra".

An R-algebra is in general not normal. However, normality is essential in the present context and will prove to hold in the two specific cases which we are going to consider. We give some equivalent formulations.

Lemma 3.7. *Let $(L; \vee, \cdot, 1)$ be an R-algebra. Then $\preccurlyeq_l$ and $\preccurlyeq_r$ are partial orders, both being extended by $\leq$. Moreover, L is normal if and only if $\preccurlyeq_l$ and $\preccurlyeq_r$ coincide.*

Proof. $\preccurlyeq_l$ and $\preccurlyeq_r$ are partial orders by (E3), the associativity property (E2), and the cancellation property (E4). By (E4) and (E3), $a \preccurlyeq_r b$ or $a \preccurlyeq_l b$ imply $a \leq b$. It is finally obvious that $\preccurlyeq_r$ and $\preccurlyeq_l$ coincide if and only if (N) holds. □

Definition 3.8. A pseudo-BCK semilattice L is called *strictly good* if we have for $a, b \in L$:

(SG) If $a \leq b$, then $(a/b)\backslash a = a/(b\backslash a)$.

For pseudo-BL algebras, this property restricted to the case that b is the minimal element, is called *good* [6]. This is the reason for our terminology.

Lemma 3.9. *Let $(L; \vee, /, \backslash, 1)$ be a divisible pseudo-BCK semilattice, and let $(L; \vee, \cdot, 1)$ be the associated R-algebra. Then L as an R-algebra is normal if and only if L as a pseudo-BCK semilattice is strictly good.*

Proof. Let L be normal. If $a \leq b$, then by by Lemma 3.3(ii), $\bar{b} = a/(b\backslash a)$ is the minimal element above b such that $a \preccurlyeq_r \bar{b}$, and $\bar{\bar{b}} = (a/b)\backslash a$ is the minimal element above b such that $a \preccurlyeq_l \bar{\bar{b}}$. But by normality, $\preccurlyeq_l = \preccurlyeq_r$, hence $\bar{b} = \bar{\bar{b}}$, that is, (SG) is fulfilled.

Conversely, let (SG) hold. If then $a \cdot b$ exists and equals c, we have $c \leq a$ and $a = c/(a\backslash c) = (c/a)\backslash c$, so $a \cdot b = c = (c/a) \cdot a$ by Lemma 3.3(i). This is one half of (N); the other one is seen similarly. □

We finally remark that (SG) implies the following property shared by integral residuated lattices which are normal in the sense of [26]; see [26, Lemma 14].

Lemma 3.10. *Let* $(L; \vee, /, \backslash, 1)$ *be a strictly good pseudo-BCK semilattice. Then, for any* $a, b \in L$*,* $a = b\backslash a$ *if and only if* $a = a/b$*.*

4 Naturally ordered R-algebras

We will now focus our attention on a class of R-algebras which is of special importance in the present context: R-algebras whose partial order is the natural one, that is, induced by the multiplication on either side. These algebras coincide with the upper-semilattice-ordered *generalized pseudoeffect algebras*, which were introduced in [13] as a double generalization of effect algebras [15].

In addition to the natural order, it will be necessary to assume one more condition, which is a version of the Riesz decomposition property and resembles the equally denoted property of po-groups (see, e.g., [20]). A list of related properties of partial algebras can be found in [11].

Definition 4.1. An R-algebra $(L; \vee, \cdot, 1)$ is called *naturally ordered* if for $a, b \in L$:

(NO) $a \leq b$ if and only if $b \cdot x = a$ for some $x \in L$ if and only if $y \cdot b = a$ for some $y \in L$.

Moreover, we say that L has the *weak Riesz decomposition property* if for $a, b, c \in L$:

(RDP_0) If $a \cdot b \leq c$, there are $a_0 \geq a$ and $b_0 \geq b$ such that $c = a_0 \cdot b_0$.

Clearly, in the presence of (NO), some of the axioms of R-algebras become redundant.

Lemma 4.2. *The partial algebra* $(L; \vee, \cdot, 1)$ *is a naturally ordered R-algebra if and only if* (E1), (E2), (E3), (E4)*, and* (NO) *hold. Moreover, a naturally ordered R-algebra is normal.*

This section is devoted to a concise proof of the fact that R-algebras subject to the two conditions of Definition 4.1 isomorphically embed into the negative cone of an ℓ-group [13]. Note that our procedure includes some optimisations when compared to the presentation in [13].

To begin with, we list some basic properties of naturally ordered R-algebras.

Lemma 4.3. *Let* $(L; \vee, \cdot, 1)$ *be a naturally ordered R-algebra. For any* $a, b, c, d \in L$*, the following holds:*

(i) *If* $a \cdot b$ *exists,* $a_1 \geq a$*, and* $b_1 \geq b$*, then also* $a_1 \cdot b_1$ *exists.*

(ii) *Let* $c \cdot a$ *and* $c \cdot b$ *exist. Then* $c \cdot (a \vee b) = (c \cdot a) \vee (c \cdot b)$.
Let $a \cdot c$ *and* $b \cdot c$ *exist. Then* $(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$.

(iii) *Let* $a \cdot b = c \cdot d$*. Then* $a \leq c$ *if and only if* $d \leq b$.

If L *fulfils* (RDP$_0$)*, we moreover have:*

(iv) *If* $a \vee b = 1$ *and if* a *and* b *possess a lower bound, then* $a \cdot b = b \cdot a = a \wedge b$.

(v) *If* $a \cdot b$ *exists and* $a \vee c = b \vee c = 1$*, then also* $a \cdot b \vee c = 1$.

Proof. Assertions (i)–(iii) are easily checked.

(iv) Let a, b, e be given such that $a \vee b = 1$ and $e \leq a, b$. So $e = x \cdot b$ for some x and, by (RDP$_0$), $a = d_1 \cdot d_2$ for some $d_1 \geq x$ and $d_2 \geq b$. But then $a, b \leq d_2$, whence $d_2 = 1$ and $x \leq a$. It follows by part (i) that $a \cdot b$ exists, and we have $e \leq a \cdot b$. Because $a \cdot b$ is a lower bound of a and b, we conclude $a \cdot b = a \wedge b$. The remaining assertion follows by symmetry.

(v) This is immediate from (RDP$_0$). □

By a scheme of the form (3) in the following lemma to hold, we mean that the product of any row and any column exists and equals the element to which the respective arrow points to; the order of multiplication is from left to right or from top to bottom, respectively.

Lemma 4.4. *Let* $(L; \vee, \cdot, 1)$ *be a naturally ordered R-algebra fulfilling* (RDP$_0$)*. Let* $a_1, \ldots, a_m, b_1, \ldots b_n \in L$ *be such that* $a_1 \cdot \ldots \cdot a_m = b_1 \cdot \ldots \cdot b_n$*, where* $n, m \geq 1$*. Then there are* $d_{11}, \ldots, d_{mn} \in L$ *such that*

$$\begin{array}{ccccc} d_{11} & \ldots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \ldots & b_n & & \end{array} \tag{3}$$

and

$$d_{ik} \vee d_{jl} = 1 \textit{ for every } 1 \leq i < j \leq m \textit{ and } 1 \leq l < k \leq n. \tag{4}$$

Proof. If $m = 1$ or $n = 1$, the assertion is trivial. Let $m = n = 2$; then our assumption is $a_1 \cdot a_2 = b_1 \cdot b_2$. Set $d_1 = a_1 \vee b_1$ and let d_2, d_3 be the unique elements such that $a_1 = d_1 \cdot d_2$ and $b_1 = d_1 \cdot d_3$.

We claim that $b_2 \leq d_2$. Indeed, by (RDP$_0$), there are $b_1 \leq e_1$ and $b_2 \leq e_2$ such that $a_1 = e_1 \cdot e_2$; and because $a_1 = d_1 \cdot d_2$ and $d_1 \leq e_1$, we have $b_2 \leq e_2 \leq d_2$ by Lemma 4.3(iii). So may choose d_4 such that $d_2 \cdot d_4 = b_2$.

By Lemma 4.3(ii), $d_1 = (d_1{\cdot}d_2) \vee (d_1{\cdot}d_3) = d_1{\cdot}(d_2 \vee d_3)$, so $d_2 \vee d_3 = 1$. By Lemma 4.3(iv) and associativity, $a_1{\cdot}a_2 = b_1{\cdot}b_2 = d_1{\cdot}d_3{\cdot}d_2{\cdot}d_4 = d_1{\cdot}d_2{\cdot}d_3{\cdot}d_4 = a_1{\cdot}d_3{\cdot}d_4$, whence $a_2 = d_3 \cdot d_4$. The proof is complete.

Assume next that $m \geq 3$ and $n \geq 2$, and that the assertion holds for any pair $m' < m$ and $n' \leq n$. It is then easily seen that it also holds for the pair m and n as well. The proof is complete. ☐

A naturally ordered R-algebra $(L; \vee, \cdot, 1)$ given, we will now construct the free semigroup with the elements of L as its generators and subject to the conditions (i) $a \cdot b = c$, where $a, b, c \in L$ such that $a \cdot b = c$ holds in L and (ii) $a \cdot b = b \cdot a$, where $a, b \in L$ such that $a \vee b = 1$ in L. The technique is, according to our knowledge, due to Baer [2]. Although the idea is simple it is not completely straightforward to see that, as a consequence of the Riesz decomposition property, the R-algebra does not "collapse" within the semigroup.

Definition 4.5. Let $(L; \vee, \cdot, 1)$ be a naturally ordered R-algebra. A sequence $(a_1, \ldots, a_n)$ of $1 \leq n < \omega$ elements of L is called a *word* of L. The set of words of L is denoted by $\mathcal{W}(L)$, and we define the product $\cdot : \mathcal{W}(L)^2 \to \mathcal{W}(L)$ as the concatenation.

Moreover, we define $\sim$ to be the smallest equivalence relation on $\mathcal{W}(L)$ such that

$$(a_1, \ldots, a_p, a_{p+1}, \ldots, a_n) \sim (a_1, \ldots, a_p \cdot a_{p+1}, \ldots, a_n)$$

and, if $a_p \vee a_{p+1} = 1$,

$$(a_1, \ldots, a_p, a_{p+1}, \ldots, a_n) \sim (a_1, \ldots, a_{p+1}, a_p, \ldots, a_n)$$

holds for any two words in $\mathcal{W}(L)$ of the indicated form, where $1 \leq p < n$. The equivalence class of some $(a_1, \ldots, a_n) \in \mathcal{W}(L)$ is denoted by $[a_1, \ldots, a_n]$, and the set of equivalence classes by $\mathcal{C}(L)$.

As seen in the next lemma, $\mathcal{C}(L)$ is a semigroup under elementwise concatenation, into which L, as a semigroup, naturally embeds.

Lemma 4.6. *Let* $(L; \vee, \cdot, 1)$ *be a naturally ordered R-algebra fulfilling* (RDP$_0$).

(i) *The equivalence relation* $\sim$ *on* $\mathcal{W}(L)$ *is compatible with* $\cdot$*.* $\cdot$ *being the induced relation,* $(\mathcal{C}(L); \cdot, [1])$ *is a monoid.*

(ii) *Let* $a_1, \ldots, a_n, b \in L$*, where* $n \geq 1$*. Then* $(a_1, \ldots, a_n) \sim (b)$ *if and only if* $a_1 \cdot \ldots \cdot a_n = b$*.*

(iii) *Let*

$$\iota \colon L \to \mathcal{C}(L), \quad a \mapsto [a]$$

be the natural embedding of L *into* $\mathcal{C}(L)$*. Then* ι *is injective.*

Furthermore, for $a, b \in L$*,* $a \cdot b$ *is defined and equals* c *if and only if* $\iota(a) \cdot \iota(b) = \iota(c)$*.*

Proof. (i) is evident.

(ii) In view of Lemma 4.3(iv), we see that for any word $(a_1, \ldots, a_n)$ the product of whose elements exists and equals b, the same is true for any word equivalent to $(a_1, \ldots, a_n)$.

(iii) The injectivity of ι follows from part (ii).

Let moreover $a, b \in L$. If $a \cdot b = c$, then obviously $\iota(a) \cdot \iota(b) = \iota(c)$. Conversely, $\iota(a) \cdot \iota(b) = \iota(c)$ means $(a, b) \sim (c)$, that is, $a \cdot b = c$ by part (ii). □

We next show that the monoid $(\mathcal{C}(L); \cdot, [1])$ fulfills the algebraic properties of the negative cone of a partially ordered group. As a preliminary, we generalize Lemma 4.4.

Lemma 4.7. *Let* $(L; \vee, \cdot, 1)$ *be a naturally ordered R-algebra fulfilling* (RDP$_0$)*. Let* $a_1, \ldots, a_m, b_1, \ldots b_n \in L$ *such that* $(a_1, \ldots, a_m) \sim (b_1, \ldots, \ldots, b_n)$*, where* $n, m \geq 1$*. Then there are* $d_{11}, \ldots, d_{mn} \in L$ *such that* (3) *and* (4) *hold.*

Proof. If $m = n$ and $a_1 = b_1, \ldots, a_m = b_n$, the assertion is trivial. Let $a_1, \ldots, b_m$ be arbitrary, and let $d_{11}, \ldots, d_{mn}$ be such that (3) and (4) hold. We shall show how to modify the scheme (3) to preserve both its correctness and the supremum-one relations (4) when $(b_1, \ldots, b_n)$ is replaced (i) by $(b_1, \ldots, b_p{\cdot}b_{p+1}, \ldots, b_n)$ for some $1 \leq p < n$, (ii) by $(b_1, \ldots, b_p^1, b_p^2, \ldots, b_n)$, where $1 \leq p \leq n$ and $b_{p1} \cdot b_{p2} = b_p$, (iii) $(b_1, \ldots, b_{p+1}, b_p, \ldots, b_n)$, where $1 \leq p < n$ and $b_p \vee b_{p+1} = 1$. The assertion will then follow.

Ad (i). We replace, for each $i = 1, \ldots, m$, the neighbouring entries d_{ip} and $d_{i,p+1}$ by their product. Then the product of the i-th row is obviously still a_i. To see that the product of the new column exists and is $b_p + b_{p+1}$, we make repeated use of Lemma 4.3(iv):

$$\begin{aligned} b_p \cdot b_{p+1} &= d_{1p} \cdot \ldots \cdot d_{mp} \cdot d_{1,p+1} \cdot \ldots \cdot d_{m,p+1} \\ &= d_{1p} \cdot d_{1,p+1} \cdot d_{2p} \cdot \ldots \cdot d_{mp} \cdot d_{2,p+1} \cdot \ldots \cdot d_{m,p+1} \\ &= \ldots \\ &= d_{1p} \cdot d_{1,p+1} \cdot d_{2p} \cdot d_{2,p+1} \cdot \ldots \cdot d_{mp} \cdot d_{m,p+1}. \end{aligned}$$

Moreover, the supremum-one relations are preserved by Lemma 4.3(v).

Ad (ii). We apply Lemma 4.4 to the equation $b_p^1 \cdot b_p^2 = d_{1p} \cdot \ldots \cdot d_{mp}$, and replace the column $d_{1p}, \ldots, d_{mp}$ with the new double column. Obviously, in the modified scheme, the rows and columns multiply correctly, and required supremum-one relations are fulfilled.

Ad (iii). We interchange the p-th and the $(p+1)$-th column. The product of the rows remains unchanged by Lemma 4.3(iv) then, and the products of the columns are the intended values. Furthermore, the newly required supremum-one relations follow from the fact that $b_p \vee b_{p+1} = 1$. □

We can now prove that $(\mathcal{C}(L); \cdot, [1])$ is a po-group cone.

Lemma 4.8. *Let* $(L; \vee, \cdot, 1)$ *be a naturally ordered R-algebra fulfilling* (RDP$_0$). *Then* $(\mathcal{C}(L); \cdot, [1])$ *is a monoid such that for* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{C}(L)$*:*

(i) *From* $\mathfrak{a} \cdot \mathfrak{b} = [1]$ *it follows* $\mathfrak{a} = \mathfrak{b} = [1]$.

(ii) *From* $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{c}$ *or* $\mathfrak{b} \cdot \mathfrak{a} = \mathfrak{c} \cdot \mathfrak{a}$ *it follows* $\mathfrak{b} = \mathfrak{c}$.

(iii) *There are* $\mathfrak{x}, \mathfrak{y} \in \mathcal{C}(L)$ *such that* $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{x} \cdot \mathfrak{a} = \mathfrak{b} \cdot \mathfrak{y}$.

Proof. The fact that $\mathcal{C}(L)$ is a monoid was the content of Lemma 4.6(i).

(i) This follows from Lemma 4.6(ii).

(ii) We may restrict to the case that $\mathfrak{a} = [a]$ for some $a \in L$. It follows from Lemma 4.6(ii) that if $\mathfrak{b} = [1]$, also $\mathfrak{b} = [1]$. Let now $\mathfrak{b} = [b_1, \ldots, b_m]$, $\mathfrak{c} = [c_1, \ldots, c_n]$, $m, n \geq 1$, and assume $(a, b_1, \ldots, b_m) \sim (a, c_1, \ldots, c_n)$. We will show that $(b_1, \ldots, b_m) \sim (c_1, \ldots, c_n)$; then the first part will follow, and the second one is proved analogously.

By Lemma 4.7, there are elements in L such that

$$\begin{array}{ccccc} d & d_1 & \dots & d_n & \to & a \\ e_1 & e_{11} & \dots & e_{1n} & \to & b_1 \\ \vdots & \vdots & & \vdots & & \vdots \\ e_m & e_{m1} & \dots & e_{mn} & \to & b_m \\ \downarrow & \downarrow & & \downarrow & & \\ a & c_1 & & c_n, & & \end{array}$$

where any pair of elements one of which is placed further up and further right than the other one, has supremum one. By (E4), $d_1 \cdot \ldots \cdot d_n = e_1 \cdot \ldots \cdot e_m$. Furthermore, by the definition of $\sim$ we may exchange successive elements in a word whenever they have supremum one; so

$$\begin{aligned} (b_1, \ldots, b_m) &\sim (e_1, e_{11}, \ldots, e_{1n}, \;.\;.\;., e_m, e_{m1}, \ldots, e_{mn}) \\ &\sim (e_1, \ldots, e_m, e_{11}, \ldots, e_{m1}, \;.\;.\;., e_{1n}, \ldots, e_{mn}) \\ &\sim (d_1, \ldots, d_n, e_{11}, \ldots, e_{m1}, \;.\;.\;., e_{1n}, \ldots, e_{mn}) \\ &\sim (d_1, e_{11}, \ldots, e_{m1}, \;.\;.\;., d_n, e_{1n}, \ldots, e_{mn}) \\ &\sim (c_1, \ldots, c_n). \end{aligned}$$

(iii) We may restrict to the case that $\mathfrak{a} = [a]$ and $\mathfrak{b} = [b]$ for some $a, b \in L$. Choose $\bar{a}, \bar{b}$ such that $\bar{a} \cdot (a \vee b) = a$ and $\bar{b} \cdot (a \vee b) = b$. Then $\bar{a} \vee \bar{b} = 1$. Now choose $\bar{\bar{a}}$ such that $\bar{\bar{a}} \cdot \bar{a} = a$. Then we have $(a, b) \sim (\bar{\bar{a}}, \bar{a}, \bar{b}, a \vee b) \sim (\bar{\bar{a}}, \bar{b}, \bar{a}, a \vee b) \sim (\bar{\bar{a}}, \bar{b}) \cdot (a)$. One half of the assertion follows; the other one is seen similarly. □

We next establish that $\mathcal{C}(L)$ is a lattice under the natural order.

Lemma 4.9. *Let* $(L; \vee, \cdot, 1)$ *be a naturally ordered R-algebra fulfilling* (RDP$_0$). *Setting*

$$\mathfrak{a} \leq \mathfrak{b} \quad \textit{if} \quad \mathfrak{b} \cdot \mathfrak{x} = \mathfrak{a} \textit{ for some } \mathfrak{x} \in \mathcal{C}(L)$$

for $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(L)$, *we endow* $\mathcal{C}(L)$ *with a partial order. We moreover have:*

(i) *Let* $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{C}(L)$ *such that* $\mathfrak{a}_1 \cdot \mathfrak{a}_2 = \mathfrak{b}_1 \cdot \mathfrak{b}_2$. *Then there are* $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4 \in \mathcal{C}(L)$ *such that*

$$\begin{array}{cccc} \mathfrak{d}_1 & \mathfrak{d}_2 & \to & \mathfrak{a}_1 \\ \mathfrak{d}_3 & \mathfrak{d}_4 & \to & \mathfrak{a}_2 \\ \downarrow & \downarrow & & \\ \mathfrak{b}_1 & \mathfrak{b}_2 & & \end{array} \qquad (5)$$

and $\mathfrak{d}_2 \vee \mathfrak{d}_3 = [1]$.

(ii) $\mathcal{C}(L)$ *is lattice-ordered.*

Proof. In view of Lemma 4.8, we easily check that $\leq$ is a partial order. Note that, by Lemma 4.8(iii), $\leq$ is actually a two-sided natural order. Moreover, the partial order on $\mathcal{C}(L)$ is directed.

(i) Let $\mathfrak{a}_1 = [a_1^1, \ldots, a_k^1]$, $\mathfrak{a}_2 = [a_1^2, \ldots, a_l^2]$, $\mathfrak{b}_1 = [b_1^1, \ldots, b_m^1]$, $\mathfrak{b}_2 = [b_1^2, \ldots, b_n^2]$, where $a_1^1, \ldots, b_n^2 \in L$. By Lemma 4.7 there are $d_{11}^1, \ldots, d_{ln}^4 \in E$ such that

$$\begin{array}{ccccccc}
d_{11}^1 & \ldots & d_{1m}^1 & d_{11}^2 & \ldots & d_{1n}^2 & \to & a_1^1 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d_{k1}^1 & \ldots & d_{km}^1 & d_{k1}^2 & \ldots & d_{kn}^2 & \to & a_k^1 \\
d_{11}^3 & \ldots & d_{1m}^3 & d_{11}^4 & \ldots & d_{1n}^4 & \to & a_1^2 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d_{l1}^3 & \ldots & d_{lm}^3 & d_{l1}^4 & \ldots & d_{ln}^4 & \to & a_l^2 \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \\
b_1^1 & \ldots & b_m^1 & b_1^2 & \ldots & b_n^2 & &
\end{array}$$

and such that every two elements in this diagram have supremum one if one of them is placed further up and further right than the other one. Define now $\mathfrak{d}_1 = [d_{11}^1, \ldots, d_{1m}^1, \ .\ .\ ., d_{k1}^1, \ldots, d_{km}^1]$ and in a similar way also $\mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4$. In view of the supremum-one relations, we conclude that the scheme (5) holds.

Let now $a \in L$ such that $\mathfrak{d}_2, \mathfrak{d}_3 \leq [a]$. We may apply Lemma 4.7 to the equivalence $(a, \ldots) \sim (d_{11}^2, \ldots, d_{kn}^2)$ to conclude that a is the product of some elements above $d_{11}^2, \ldots, d_{kn}^2$. But by the same reasoning we see that a is also the product of elements above $d_{11}^3, \ldots, d_{lm}^3$. So $a = 1$, and it follows that $\mathfrak{d}_2 \vee \mathfrak{d}_3 = [1]$ in the poset $\mathcal{C}(L)$.

(ii) Let $\mathfrak{a}_1, \mathfrak{b}_1 \in \mathcal{C}(L)$. By the directedness, there are $\mathfrak{a}_2, \mathfrak{b}_2$ such that $\mathfrak{a}_1 \cdot \mathfrak{a}_2 = \mathfrak{b}_1 \cdot \mathfrak{b}_2$. By part (i), there are $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4$ such that (5) and $\mathfrak{d}_2 \vee \mathfrak{d}_3 = [1]$ holds. We show that $\mathfrak{d}_1 \cdot \mathfrak{d}_2 \cdot \mathfrak{d}_3 = \mathfrak{a}_1 \wedge \mathfrak{b}_1$. Indeed, $\mathfrak{d}_1 \cdot \mathfrak{d}_2 \cdot \mathfrak{d}_3 \leq \mathfrak{a}_1, \mathfrak{b}_1$, and if $\mathfrak{e} \leq \mathfrak{a}_1, \mathfrak{b}_1$, we have $\mathfrak{e} = \mathfrak{a}_1 \cdot \mathfrak{f}_1 = \mathfrak{b}_1 \cdot \mathfrak{f}_2$ and by Lemma 4.8(ii) $\mathfrak{d}_2 \cdot \mathfrak{f}_1 = \mathfrak{d}_3 \cdot \mathfrak{f}_2$ for some $\mathfrak{f}_1, \mathfrak{f}_2$. Applying part (i) again, we conclude $\mathfrak{e} \leq \mathfrak{d}_1 \cdot \mathfrak{d}_2 \cdot \mathfrak{d}_3$.

Furthermore, we show that $\mathfrak{d}_1 = \mathfrak{a}_1 \vee \mathfrak{b}_1$. Clearly, $\mathfrak{a}_1, \mathfrak{b}_1 \leq \mathfrak{d}_1$. Let $\mathfrak{e} \geq \mathfrak{a}_1, \mathfrak{b}_1$. Choose $\mathfrak{f}$ such that $\mathfrak{e} \cdot \mathfrak{f} = \mathfrak{a}_1 \cdot \mathfrak{a}_2$. Then $\mathfrak{f} \leq \mathfrak{a}_2, \mathfrak{b}_2$. By similar reasoning as in the preceding paragraph, we conclude $\mathfrak{a}_2 \wedge \mathfrak{b}_2 = \mathfrak{d}_2 \cdot \mathfrak{d}_3 \cdot \mathfrak{d}_4$. It follows $\mathfrak{d}_1 \leq \mathfrak{e}$. □

We arrive at our main theorem. By an isomorphic embedding of a naturally ordered R-algebra $(L; \vee, \cdot, 1)$ into an ℓ-group $(G; \wedge, \vee, \cdot, 1)$, we mean an injective mapping

$\iota\colon\ L \to G$ such that for $a, b, c \in L$

$$\begin{aligned} \iota(a \vee b) &= \iota(a) \vee \iota(b), \\ a \cdot b \text{ is defined and equals } c &\text{ if and only if } \iota(a) \cdot \iota(b) = \iota(c), \\ \iota(1) &= 1. \end{aligned} \tag{6}$$

Note that, with respect to the partial product on L, ι is what is called a full homomorphism in [21, §13]. Namely, a product is defined in L exactly if the product of the images is inside the range of ι.

Theorem 4.10. *Let $(L; \vee, \cdot, 1)$ be a naturally ordered R-algebra fulfilling* (RDP$_0$). *Then there is an isomorphic embedding ι of the R-algebra $(L; \vee, \cdot, 1)$ into the ℓ-group $(\mathcal{G}(L); \wedge, \vee, \cdot, [1])$. The range of ι is a convex subset of $\mathcal{G}(L)$, upper-bounded by $[1]$, which generates $\mathcal{G}(L)$.*

Proof. By a theorem of Birkhoff (see [16, Theorem II.4]), Lemma 4.8 implies that there is a po-group $\mathcal{G}(L)$ such that $\mathcal{C}(L) = \mathcal{G}(L)^-$ and $\mathcal{G}(L) = \mathcal{C}(L) \cdot \mathcal{C}(L)^{-1} = \mathcal{C}(L)^{-1} \cdot \mathcal{C}(L)$. By Lemma 4.9(ii), it furthermore follows that $\mathcal{G}(L)$ is lattice-ordered.

Clearly, we take $\iota :\ L \to \mathcal{G}(L),\ \ a \mapsto [a]$. We have $\iota(1) = [1]$. Furthermore, by Lemma 4.6(iii), ι is injective.

By the same lemma, $a \cdot b = c$ in L if and only if $\iota(a) \cdot \iota(b) = \iota(c)$ in $\mathcal{C}(L)$.

By construction, $\iota(L)$ generates $\mathcal{G}(L)$ as a group. We next show that $\iota(L)$ is a convex subset of $\mathcal{C}(L) = \mathcal{G}(L)^-$ containing $[1]$. Let $\mathfrak{a} \in \mathcal{G}(L)$ and $b \in L$ such that $[b] \leq \mathfrak{a} \leq [1]$. Then $\mathfrak{a} \cdot \mathfrak{c} = [b]$ for some $\mathfrak{c} \in \mathcal{C}(L)$, and by Lemma 4.6(ii) it follows that $\mathfrak{a} = \iota(a)$ for some $a \in L$.

Finally, let $a, b \in L$. If $a \leq b$, clearly $\iota(a) \leq \iota(b)$. Conversely, if $\iota(a) \leq \iota(b)$, then $[b] \cdot \mathfrak{c} = [a]$ for some $\mathfrak{c} \in \mathcal{C}(L)$, and we conclude by Lemma 4.6(ii) that $b \cdot c = a$ for some $c \in L$, that is, $a \leq b$. The preservation of finite suprema is now easily derivable. ☐

In the sequel, we will consider a partial algebra L as a subset of the group G representing L according to Theorem 4.10. Note that then the partial multiplication on L is simply the restriction of the multiplication in G to those pairs of elements of G whose product is again an element of L.

5 Cone algebras

We are going to apply the representation theorem 4.10 to two classes of pseudo-BCK semilattices. The first one was proposed by B. Bosbach [4].

Definition 5.1. A divisible pseudo-BCK semilattice is called a *cone algebra* if, for any $a, b \in L$,

(B) $a \vee b \;=\; a/(b\backslash a) = (a/b)\backslash a$.

The original definition in [4] was certainly different, but can be seen equivalent by the subsequent lemma. We note furthermore that in [14], the notion *pseudo-ŁBCK-algebra* was used.

Lemma 5.2. *The algebra* $(L; /, \backslash)$ *is the* $/, \backslash$*-reduct of a cone algebra if and only if, for any* $a, b, c \in L$*,* (i) $(c/a)/(b/a) = (c/b)/(a/b)$ *and* $(a\backslash b)\backslash(a\backslash c) = (b\backslash a)\backslash(b\backslash c)$*,* (ii) $(a\backslash a)\backslash b = b/(a/a) = b$*,* (iii) $c\backslash(a/b) = (c\backslash a)/b$*, and* $a/(b\backslash a) = (b/a)\backslash b$*.*

In this case, the algebra can be expanded in a unique way to a cone algebra, putting $1 = a\backslash a$ *for an arbitrary* a*, and defining the partial order by* $a \leq b$ *if* $a\backslash b = 1$*.*

Proof. For the indicated properties of cone algebras, see Section 2. For the other direction, we refer to [4]. □

Lemma 5.3. *Let* $(L; \vee, \cdot, 1)$ *be the R-algebra associated to a cone algebra. Then* L *is a naturally ordered R-algebra fulfilling* (RDP_0).

Proof. If, for $a, b \in L$, $a \leq b$, then $b = a/(b\backslash a) = (a/b)\backslash a$ by (B), so L is a naturally ordered R-algebra.

To see (RDP_0), let $a, b, c \in L$ such that $a \cdot b \leq c$. Let $a_1 = a \vee c$ and choose b_1 such that $c = a_1 \cdot b_1$. Clearly, $a \leq a_1$. Furthermore, from $c/(a\backslash c) \geq a \vee c = a_1$ we conclude $a\backslash c_1 = a\backslash c$; it follows $b_1 = a\backslash c_1 = a\backslash c \geq a\backslash(a \cdot b) = b$. □

So we have proved the following representation theorem. Using completely different techniques, this theorem was first proved by Bosbach [4]. A proof along the lines of the present paper is contained in [14]. A further proof of this theorem can be found in [17].

Theorem 5.4. *Let $(L; \vee, /, \backslash, 1)$ be a cone algebra. Then there is an ℓ-group $(G; \wedge, \vee, \cdot, 1)$ such that $(L; \vee, 1)$ is a convex subalgebra of $(G^-; \vee, 1)$, L generates G as a group, and for $a, b \in L$*

$$\begin{aligned} a/b &= a \cdot b^{-1} \wedge 1, \\ b\backslash a &= b^{-1} \cdot a \wedge 1. \end{aligned} \tag{7}$$

Proof. By Theorem 4.10, the associated R-algebra $(L; \vee, \cdot, 1)$ isomorphically embeds into the ℓ-group $(\mathcal{G}(L), \wedge, \vee, \cdot, 1)$. For $a, b \in L$, we have $a = (a \vee b) \cdot (b\backslash a) = (a/b) \cdot (a \vee b)$ by (B) and Lemma 3.3(i); (7) follows. ☐

6 Divisible pseudo-BCK algebras: the linear case

A pseudo-BCK semilattice is called *semilinear* (or *representable*) if it is the subdirect product of pseudo-BCK semilattices whose order is linear. The class consisting of these algebras forms a variety [30]. In order to describe the semilinear divisible pseudo-BCK semilattices, the case of a linear order needs to be considered, and this is what we shall do in this section.

Definition 6.1. A *pseudo-BCK toset* is a pseudo-BCK semilattice whose partial order is linear.

We will show that divisible pseudo-BCK tosets correspond to the ordinal sum of naturally ordered R-algebras, which in turn are representable by linearly ordered groups according to Theorem 4.10.

We give in this way an alternative proof of the representation theorem in [10].

The only difficult step is the proof that pseudo-BCK tosets are strictly good.

Lemma 6.2. *Let $(L; \vee, /, \backslash, 1)$ be a divisible pseudo-BCK toset. Let $(L; \vee, \cdot, 1)$ be the associated R-algebra.*

(i) *Let $a \leq b \leq c < 1$. Then $a \preccurlyeq_r b$ and $b \preccurlyeq_r c$ if and only if $a \preccurlyeq_r c$. Similarly, $a \preccurlyeq_l b$ and $b \preccurlyeq_l c$ if and only if $a \preccurlyeq_l c$.*

(ii) *Let $a \leq b < 1$. Then $a \preccurlyeq_r b$ if and only if $a \preccurlyeq_l b$.*

Proof. (i) $a \preccurlyeq_r b$ and $b \preccurlyeq_r c$ imply $a \preccurlyeq_r c$ by the axiom of associativity, (E2).

Assume $a \preccurlyeq_r c$. Then $b \preccurlyeq_r c$ follows by (E6). Moreover, we have $c = b/(c\backslash b) \leq [a/(b\backslash a)]/(c\backslash b) = a/(c\backslash a) = c$ by $b \preccurlyeq_r c$, Lemma 2.4(viii), and $a \preccurlyeq_r c$. So

$[a/(b\backslash a)]/(c\backslash b) = b/(c\backslash b) < 1$. In view of the linear order, it follows $c\backslash b > b$ and $c\backslash b > a/(b\backslash a)$. So by Lemma 2.4(vii), $b = a/(b\backslash a)$, that is, $a \preccurlyeq_r b$.

The proof of the first half is complete; the second half is shown analogously.

(ii) Let $a \preccurlyeq_r b$. We will show $a \preccurlyeq_l b$; the converse direction will follow analogously.

Let x be such that $b \cdot x = a$. If $b \leq x$, then $a \preccurlyeq_l x$ and $a \leq b \leq x$ imply $a \preccurlyeq_l b$ by part (i).

So let us assume $b > x$. Then $a \preccurlyeq_r b$ and $a \leq x \leq b$ imply $a \preccurlyeq_r x$ by part (i); let s be such that $x \cdot s = a$. If then $b \leq s$, we conclude $a \preccurlyeq_l b$ from $a \preccurlyeq_l s$.

Let us assume $b > s$. Since then $b \cdot x = x \cdot s < s < b$, there is by (E6) an $x \leq r$ such that $s = b \cdot r$. So $a = b \cdot x = x \cdot b \cdot r$ then.

If $b \leq r$, then $a \preccurlyeq_l r$ implies $a \preccurlyeq_l b$. If $b > r$, then $x \cdot b < r < b$ implies $r = u \cdot b$ for some u, hence $a = x \cdot b \cdot u \cdot b$, that is, $a \preccurlyeq_l b$. □

In view of Lemmas 3.7 and 3.9, part (ii) of this lemma shows that divisible pseudo-BCK tosets are strictly good and the associated R-algebras are normal.

We note that for the natural question if all divisible pseudo-BCK semilattices are strictly good, recently the negative answer was given [9]; there are pseudo-BL algebras which are not good. So in particular, property (SG) is not redundant.

We furthermore note that the proof of part (i) of Lemma 6.2 is a significant improvement of the proof which we have given in the commutative case in [37].

Definition 6.3. Let $(I; \leq)$ be a linearly ordered set, and for every $i \in I$, let $(L_i; \vee, \cdot, 1_i)$ be an R-algebra. Put $L = \dot{\bigcup}_{i \in I}(L_i \backslash \{1_i\}) \cup \{1\}$, where 1 is a new element. For $a, b \in L$, put $a \leq b$ if either $b = 1$, or $a \in L_i$ and $b \in L_j$ such that $i < j$, or $a, b \in L_i$ for some i and $a \leq b$ holds in L_i. Similarly, define $a \cdot b$ if either $a = 1$, in which case $a \cdot b = b$, or $b = 1$, in which case $a \cdot b = a$, or $a, b \in L_i$ for some i and $a \cdot b$ is defined in L_i, in which case $a \cdot b$ is mapped to the same value as in L_i. Then $(L; \vee, \cdot, 1)$ is called the *ordinal sum* of the R-algebras L_i w.r.t. $(I; \leq)$.

Lemma 6.4. *An ordinal sum of R-algebras is again an R-algebra.*

An ordinal sum of normal R-algebras is again a normal R-algebra.

Lemma 6.5. *Let $(L; \vee, /, \backslash, 1)$ be a divisible pseudo-BCK toset. Let $(L; \vee, \cdot, 1)$ be the associated R-algebra. Then L is the ordinal sum of linearly and naturally ordered R-algebras fulfilling* (RDP_0).

Proof. By Lemma 6.2(i), $L\backslash\{1\}$ is the disjoint union of convex subsets C such that

$a \preccurlyeq_r b$ for any $a, b \in C$ such that $a \leq b$, but not for any pair of elements different from one and from distinct subsets.

Let C one of these subsets, and consider $C \cup \{1\}$ endowed with the restriction of $\leq$ and $\cdot$ to $C \cup \{1\}$ as well as with the constant 1. Then $(C \cup \{1\}; \vee, \cdot, 1)$ fulfills the axioms (E1)–(E4) and, by Lemma 6.2(ii), also (NO). So $C \cup \{1\}$ is a naturally ordered R-algebra. Since $C \cup \{1\}$ is linearly ordered, also (RDP_0) holds. ☐

In view of Theorem 4.10, we have proved the following representation theorem.

Theorem 6.6. *Let $(L; \vee, /, \backslash, 1)$ be a divisible pseudo-BCK toset. Then $L\backslash\{1\}$ is the disjoint union of convex subsets C_i, $i \in I$, such that the following holds. For each i, there is a linearly ordered group $(G_i; \wedge, \vee, \cdot, 1)$ such that $(C_i \cup \{1\}; \vee, 1)$ is a convex subalgebra of $(G_i^-; \vee, 1)$ and $C_i \cup \{1\}$ generates G_i as a group. Moreover, if $a, b \in C_i$ for some $i \in I$,*

$$a/b = \begin{cases} a \cdot b^{-1} & \text{if } a < b, \\ 1 & \text{otherwise,} \end{cases} \qquad b\backslash a = \begin{cases} b^{-1} \cdot a & \text{if } a < b, \\ 1 & \text{otherwise,} \end{cases}$$

where the differences are calculated in G_i. If $a \in C_i$ and $b \in C_j$ such that $i \neq j$,

$$a/b \;=\; b\backslash a \;=\; \begin{cases} a & \text{if } a < b, \\ 1 & \text{if } b < a. \end{cases}$$

Note how that the sets C_i are, apart from the 1, "initial pieces" of the linearly ordered groups G_i. C_i can be of the form $G_i^- \backslash \{1\}$ or $\{g \in G_i \colon \; u \leq g < 1\}$ for some $u < 1$, as known from the case of pseudohoops. In addition, C_i can also be of the form $\{g \in G_i \colon \; u < g < 1\}$ for some $u < 1$, or C_i is bounded from below, but does not possess an infimum in G_i.

7 Pseudo-BCK semilattices and residuated lattices

Although pseudo-BCK algebras were introduced independently from residuated lattices, both notions are closely related. For an overview concerning residuated lattices, see [27]; for a comprehensible account, we refer to [18]. For the particular notion of divisibility which is central in this paper, see [25, 26].

Definition 7.1. An *integral residuated lattice* is an algebra $(L; \wedge, \vee, \circ, /, \backslash, 1)$ such that (P1) $(L; \wedge, \vee, 1)$ is a upper-bounded lattice, (P2) $(L; \circ, 1)$ is monoid, and (P3) for any $a, b, c \in L$,

$$a \circ b \leq c \;\text{ if and only if }\; b \leq a\backslash c \;\text{ if and only if }\; a \leq c/b. \tag{8}$$

The close relationship between pseudo-BCK semilattices and integral residuated lattices is, for the general case, expressed by the subsequent theorem of J. Kühr ([28], see also [29]), which was independently found also by C. van Alten [36]. The intention of this section is to use the representation theorems 5.4 and 6.6 to specialize this relationship to the two subclasses of pseudo-BCK semilattices considered in this paper.

Theorem 7.2. *Let* $(L; \wedge, \vee, \circ, /, \backslash, 1)$ *be an integral residuated lattice, and let* $(L'; \vee, /, \backslash, 1)$ *be a subalgebra of its* $\vee, /, \backslash, 1$*-reduct. Then* L' *is a pseudo-BCK semilattice.*

All pseudo-BCK semilattices are of this form.

So in particular, the $\vee, /, \backslash, 1$-reduct of an integral residuated lattice is a pseudo-BCK semilattice. Furthermore, this reduct determines the residuated lattice uniquely. So we may wonder under which conditions a pseudo-BCK semilattice of a type considered in this paper is expandable to a residuated lattice. In the remaining cases, a pseudo-BCK semilattice is by Theorem 7.2 a proper subalgebra of such a reduct; we may wonder if this reduct can always be chosen to belong to the same class.

We will first consider the case of cone algebras. The corresponding subclass of residuated lattices is the following [3].

Definition 7.3. An integral residuated lattice such that (B) holds is called a *GMV-algebra*.

We note that this is what in certain papers is referred to as an "integral GMV-algebra".

Proposition 7.4. *Let* $(L; \wedge, \vee, \circ, /, \backslash, 1)$ *be an integral residuated lattice. Then* L *is a GMV-algebra if and only if* $(L; \vee, /, \backslash, 1)$ *is a cone algebra.*

Proof. Let L be a GMV-algebra. Then the $\vee, /, \backslash, 1$-reduct is a pseudo-BCK semilattice by Theorem 7.2. L fulfils (B) by assumption. For a proof of the divisibility conditions (D), see [3, Proposition 5.1] and the remarks after Definition 2.2. One direction of the asserted equivalence follows; the other one is clear. □

So given a cone algebra L, the question is when L is the reduct of a GMV-algebra and if L is otherwise a subreduct of a GMV-algebra. The easy answer is found on the basis of the representation by Theorem 5.4.

Theorem 7.5. *Let* $(L;\vee,/,\backslash,1)$ *be a cone algebra, and let* $(G;\wedge,\vee,\cdot,1)$ *be the* ℓ*-group containing* L *in accordance with Theorem* 5.4 *and the notation used there. Then* L *is the reduct of a GMV-algebra if and only if* (i) L *is lattice-ordered and* (ii) *for any* $a,b\in L$ *there is a minimal element* $c\in L$ *such that* $a\cdot b\leq c$*.*

In any case, $(L;\vee,/,\backslash,1)$ *is a subreduct of the GMV-algebra* $(G^{-};\wedge,\vee,\cdot,\backslash,/,1)$*, where* $\backslash$ *and* $/$ *are defined by* (7).

Proof. By Proposition 7.4, L is the reduct of a GMV-algebra if and only if L is expandable to an integral residuated lattice. This in turn is the case if and only if L is lattice-ordered and, for any $a,b\in L$, the set $\{c\in L:\ b\leq a\backslash c\}$ contains a minimal element. The latter condition holds iff, for any $a,b\in L$, the set $\{c\in L:\ b\leq (a\vee c)^{-1}\cdot c\}=\{c\in L:\ a\cdot b\leq c\}$ has a minimum, that is, if there is a minimal element in L above $a\cdot b$. The first assertion is proved. The second assertion is obvious. □

From this theorem, we easily derive a representation theorem for GMV-algebras, an elegant formulation of which can be found in [17].

We will next consider the case of divisible pseudo-BCK tosets.

Definition 7.6. An integral residuated lattice is called *divisible* if for $a,b\in L$:

(D') $a\wedge b\ =\ (a/b)\circ b\ =\ b\circ(b\backslash a)$.

We note that this notion comes very close to the notion of a pseudohoop, but is not the same; however, in the case of a linear order, the two notions do coincide.

Proposition 7.7. *Let* $(L;\wedge,\vee,\circ,/,\backslash,1)$ *be a semilinear integral residuated lattice. Then* L *is divisible if and only if* $(L;\vee,/,\backslash,1)$ *is a divisible pseudo-BCK semilattice.*

Proof. Let L fulfil (D'). For the derivation of (D), see the remarks after Definition 2.2.

Conversely, let $(L;\vee,/,\backslash,1)$ be a divisible pseudo-BCK semilattice. Let $a,b\in L$. From (D), we have $(a\circ(a\backslash b))\backslash c=(a\backslash b)\backslash(a\backslash c)=(b\backslash a)\backslash(b\backslash c)=(b\circ(b\backslash a))\backslash c$ for any c, and it follows $a\circ(a\backslash b)=b\circ(b\backslash a)$. Similarly, we conclude $b\circ(b\backslash a)=(b/a)\circ a$.

We have $a\circ(a\backslash b)\leq a,b$. If now $d\leq a,b$, it follows by the semilinearity that $d=d\circ((b\backslash a)\vee(a\backslash b))=(d\circ(b\backslash a))\vee(d\circ(a\backslash b))\leq a\circ(a\backslash b)$. So $a\wedge b=a\circ(a\backslash b)$, and the proof of (D') is complete. □

So we pose again the question which divisible pseudo-BCK tosets are reducts and which are proper subreducts of divisible integral residuated lattices. As before, the answer is found on the basis of the representation by Theorem 6.6. The ordinal sum of integral residuated lattices is defined like in [1].

Theorem 7.8. *Let* $(L; \vee, /, \backslash, 1)$ *be a divisible pseudo-BCK toset. Let* C_i*,* $i \in I$*, be the convex subsets of* L *and* $(G_i; \vee, \cdot, 1)$*,* $i \in I$*, the linearly ordered groups containing* $C_i \cup \{1\}$ *in accordance with Theorem* 6.6 *and the notation used there. Then* L *is the reduct of a divisible integral residuated lattice if and only if, for each* i*, either* C_i *possesses a minimal element or is in* G_i *not bounded from below.*

In any case, the ordinal sum of the GMV-algebras $(G_i^-; \wedge, \vee, \cdot, /, \backslash, 1)$*,* $i \in I$*, is a divisible integral residuated lattice, of which* $(L; \vee, /, \backslash, 1)$ *is a subreduct.*

Proof. By Proposition 7.7, L is the reduct of a divisible integral residuated lattice if and only if L is expandable to an integral residuated lattice. This in turn is the case if and only if, for any $a, b \in L$, the set $\{c \in L : b \leq a \backslash c\}$ possesses a minimum. If $a = 1$ or $b = 1$ or if $a \in C_i$ and $b \in C_j$, where $i \neq j$, this is the case. Let $a, b \in C_i$ for some $i \in I$. If then $C_i \cup \{1\} = G_i^-$, the minimum is $a \cdot b$; if then $C_i \cup \{1\} = [u_i, 1]$ for some $u_i \in G_i^- \backslash \{1\}$, the minimum is $a \cdot b \vee u_i$. However, if C_i is in G_i bounded from below, but does not possess a minimal element, choose $a, b \in C_i$ such that $a \cdot b < c$ for all $c \in C_i$; then the minimum does not exist. The proof of the first part is complete. The last claim is obvious. □

From this theorem, we easily derive the representation theorem of A. Dvurečenskij [8] on pseudohoops.

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