t-norms induced by metrics on boolean algebras

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Abstract

Let d_ν be the metric associated with a strictly positive submeasure ν on some boolean algebra $\mathcal{P}$. If d_ν is bounded from above by 1, $E_\nu = 1 - d_\nu$ is a (fuzzy) similarity relation on $\mathcal{P}$ at least w.r.t. the Łukasiewicz t-norm, but possibly also w.r.t. numerous further t-norms.

In this paper, we show that under certain assumptions on $\mathcal{P}$ and ν, we may associate with ν in a natural way a continuous t-norm w.r.t. which E_ν is a similarity relation and which, in a certain sense, is the weakest such t-norm. Up to isomorphism, every continuous t-norm arises in this way.

Keywords: continuous triangular norms, boolean algebras endowed with a metric

1 Introduction

Triangular norms - or t-norms for short - might have a complicated structure, which in the general case is actually not yet understood. In the case of continuity, we know that there is still an abundance of possibilities; such a t-norm is the ordinal sum of countably many copies of the three standard t-norms, where we have a practically free choice of how to put the components into a linear order. We wonder if there might be any practical reason to deal with such constructs. Besides, we actually wonder about the easy cases as well; we do not yet find the question satisfactorily answered in the literature in which prototypical situation one of the three standard t-norms is the natural choice.

This paper is motivated by the question if a specific continuous t-norm, possibly resulting from a complicated construction, may be of practical use. It is our intention to contribute to a positive answer by pointing out a correspondence between continuous t-norms on the one hand and certain metric boolean algebras on the other hand.

Inspired by U. Höhle's paper [Hoe], we shall argue as follows. Let an boolean algebra $\mathcal{P}$ be given which is endowed with a metric d. Let us think of $\mathcal{P}$ as representing a system of sharp propositions arising in some context, and assume that d tells to what extent two propositions differ from each other. Now, if d is bounded from above by 1, the transition from d to the function $E = 1 - d$ allows an interpretation in a way common in fuzzy logics: $E(a, b)$ tells how similar two elements $a, b \in \mathcal{P}$ are, 1 meaning coincidence of a and b, and 0 meaning that a and b have nothing in common. Furthermore, given a t-norm $\odot$, E may happen to be a similarity relation on $\mathcal{P}$ with respect to $\odot$. In particular, E is always a similarity relation w.r.t. the Łukasiewicz t-norm; and moreover, if e.g. ν is an ultrametric, E is also a similarity relation w.r.t. the minimum t-norm [Hoe, Section 3.2].

So the following question arises. Given a metric d on a space $\mathcal{P}$, is there a reason to associate with d a *canonical* t-norm w.r.t. which $E = 1 - d$ is a similarity relation? In particular, is there a weakest such t-norm? This is the problem studied in this paper.

Our results may be summarized as follows. Our base space will be a boolean algebra of a particularly easy form: the algebra $\mathcal{P}$ of subsets of an at most countable set. To examine the general case remains as a task for future work. Furthermore, it seems natural to assume that a metric on $\mathcal{P}$ is the metric d_ν associated with some submeasure ν; see e.g. [Fre2]. We will do so, and in addition, we will impose several conditions on this submeasure. The crucial property is what we call homogeneity; accordingly, $\nu(a) \leq \nu(b \vee c)$ implies the existence of elements $\bar{b}$ and $\bar{c}$ such that $a \leq \bar{b} \vee \bar{c}$, where $\nu(\bar{b}) = \nu(b)$ and $\nu(\bar{c}) = \nu(c)$. We then proceed to describe the exact structure of the submeasures we deal with. Interestingly, this analysis shows strong analogies to the analysis of continuous t-norms.

We continue showing that the metric d_ν associated with such a submeasure ν determines a specific continuous t-norm up to isomorphism. Namely, with respect to the values the fuzzy relation $E_\nu = 1 - d_\nu$ can take, it is the weakest (pointwise maximal) one among those t-norms w.r.t. which E_ν is a similarity relation. Our construction will show that, up to isomorphism, every continuous t-norms is determined by an appropriately chosen metric d_ν.

2 Metrics on boolean algebras

We deal in this section with boolean algebras endowed with a metric. Intuitively, the elements of this algebra should be thought of as representing certain sharp propositions which arise simultanously in some context; the metric then tells us to what degree two such propositions differ. Similarity relations associated with metrics will be discussed only in the subsequent Section 3.

As usual in this context, we will assume that the metric is induced by some submeasure on the boolean algebra. Accordingly, the metrical distance between a pair of elements is the value which the submeasure assigns to the symmetric difference of this pair. Moreover, we will postulate that the submeasure fulfils several conditions, among which the so-called homogeneity is most remarkable. This condition is unusual in that it refers to elements which are possibly algebraically unrelated.

For a general discussion of submeasures on boolean algebras and metrics associated to them, we refer to [Fre1, Section 5.5]. We should stress, however, that our concerns are different from those in [Fre1] and other papers. In particular, typical properties of submeasures like exhaustiveness, continuity and the like are in general not shared by the submeasures discussed here.

Definition 2.1 Let $(\mathcal{P}; \leq, \neg, 0, 1)$ a σ-complete boolean algebra, and let $[0, 1]$ be the real unit interval.

(i) $\nu\colon\ \mathcal{P} \to \mathbb{R}^+$ is called a *submeasure* if (α) $\nu(0) = 0$, (β) $a \leq b$

implies $\nu(a) \leq \nu(b)$, and (γ) $\nu(a \vee b) \leq \nu(a) + \nu(b)$.

A submeasure ν is *strictly positive* if $\nu(a) = 0$ only in case $a = 0$; and ν is *normalized* if $\nu(1) = 1$. We shall call ν *normal* if ν is strictly positive and normalized.

ν is called *continuous from below* if $b_0 \leq b_1 \leq \ldots$ and $a = \bigvee_i b_i$ imply $\nu(a) = \bigvee_i \nu(b_i)$.

Furthermore, ν is called *homogeneous* if for any $a, b_0, b_1, \ldots$ such that $\nu(a) \leq \nu(\bigvee_i b_i)$, there are $\bar{b}_0, \bar{b}_1, \ldots$ such that $a \leq \bigvee_i \bar{b}_i$ and, for every i, $\nu(\bar{b}_i) = \nu(b_i)$.

Finally, ν is called *faithful* if for any $a \in \mathcal{P}$ which is not an atom, there is a $b < a$ such that $0 < \nu(b) < \nu(a)$.

(ii) Let ν be a normal submeasure on $\mathcal{P}$. Then

$$d_\nu \colon \mathcal{P} \times \mathcal{P} \to [0, 1], \quad (a, b) \mapsto \nu(a \triangle b)$$

is called the *metric on $\mathcal{P}$ induced by ν*. Here, $a \triangle b = (a \setminus b) \cup (b \setminus a)$ is the symmetric difference of a and b.

For some submeasure ν, we will occasionally express the fact that $\nu(a) = v$ by saying that *a is of measure v*.

It is easy to verify that the metric on a boolean algebra induced by a normal submeasure is indeed a metric; cf. [Fre2, Lemma 393B].

We further note that the condition of homogeneity of a submeasure on a boolean algebra may be split into two parts.

Lemma 2.2 *Let $\nu \colon \mathcal{P} \to [0, 1]$ be a submeasure on a complete boolean algebra $\mathcal{P}$. Then ν is homogeneous if and only if the following two conditions hold:*

(i) *For any $a, b_0, b_1, \ldots$ such that $\nu(a) \leq \nu(\bigvee_i b_i)$, there are $\bar{b}_0, \bar{b}_1, \ldots$ such that $a = \bigvee_i \bar{b}_i$ and, for every i, $\nu(\bar{b}_i) \leq \nu(b_i)$.*

(ii) *For any a, b such that $\nu(a) \leq \nu(b)$ there is a $\bar{b} \geq a$ such that $\nu(\bar{b}) = \nu(b)$.*

We will see that the conditions enumerated in Definition 2.1(i) enable us to describe completely the structure of submeasures. To that end, however, we have to assume that the underlying boolean algebra is of a particularly easy form, namely that it is atomistic and complete. Although it surely would be desirable to drop these assumptions, we will see that the aim of this paper, to motivate the usage of any specific continuous t-norm, is still achievable.

For an element a of a complete atomistic boolean algebra $\mathcal{P}$, we will denote by $\mathcal{A}(a)$ the set of atoms below a. We may then identify $\mathcal{P}$ with the algebra of subsets of $\mathcal{A}(1)$. We note that we could express all what follows also in set-theoretical terms; however, the reason to prefer the algebraic notation is to support our intuition that we have to do with a system of propositions and also to simplify consistency with future works.

So the rest of this section is devoted to the characterisation of normal submeasures which are continuous from below, homogeneous, and faithful. We start with the simplest way of how to construct submeasures.

In the sequel, $\operatorname{card} X$ denotes the cardinality of some set X. Recall that a boolean algebra is separable if it is countably generated.

Definition 2.3 Let $(\mathcal{P}; \leq, \neg, 0, 1)$ be an atomistic, separable, σ-complete boolean algebra. If then $n = \operatorname{card} \mathcal{A}(1)$ is finite, let $N = \{0, 1, \ldots, n\}$, else $N = \mathbb{N}$. Let $s\colon\ N \to [0, 1]$ be such that (α) $s(0) = 0$, (β) $k < l$ implies $s(k) < s(l)$ for $k, l \in N$, (γ) $s(k + l) \leq s(k) + s(l)$ for $k, l \in N$ such that $k + l \in N$, and (δ) $\bigvee\{s(k)\colon\ k \in N\} = 1$. Then we call

$$\sigma\colon\ \mathcal{P} \to [0, 1], \quad a \mapsto \begin{cases} s(\operatorname{card} \mathcal{A}(a)) & \text{if } \mathcal{A}(a) \text{ is finite,} \\ 1 & \text{else.} \end{cases}$$

a *standard submeasure* on $\mathcal{P}$.

Note that a standard submeasure is indeed a submeasure. Its intuitive meaning is clear: It is nothing but the counting measure whose range, an initial piece of the natural numbers plus infinity, is squeezed into the real unit interval, in a way probably giving up additivity, but preserving subadditivity.

The following four examples will illustrate our further analysis. Already here, the structural similarity of certain submeasures on the one hand and continuous t-norms on the other hand, is clearly seen.

Example 2.4 Let X be a countable set, and let $\mathcal{P}$ be the boolean algebra of subsets of X. We shall give four examples of a submeasure ν: $\mathcal{P} \to [0,1]$. Let $A \subseteq X$.

1. 1 Let X be finite; let X contain n elements. Define

$$\nu(A) = \frac{\operatorname{card} A}{n}.$$

2. 2 Let X be countably infinite. Define

$$\nu(A) = \begin{cases} 1 - \frac{1}{2^m} & \text{if } m = \operatorname{card} A < \infty \\ 1 & \text{if } A \text{ is infinite.} \end{cases}$$

3. 3 Let $X = \{x_i\colon\ i < \lambda\}$, where $1 \leq \lambda \leq \omega$, and let $m_i \in \mathbb{R}$, $i < \lambda$, such that $0 < m_1 < m_2 < \ldots$ and $\bigvee_i m_i = 1$. Define

$$\nu(A) = \bigvee\{m_i\colon\ a_i \in A\}.$$

4. 4 Let $X = Y \cup Z$, where $Y = \{y_1, \ldots, y_m\}$ and $Z = \{z_1, \ldots, z_n\}$ are finite disjoint sets. Define

$$\nu(A) = \begin{cases} \frac{\operatorname{card} A}{2m} & \text{if } A \cap Y = \emptyset \\ \frac{1}{2} + \frac{\operatorname{card}(A \cap Z)}{2n} & \text{else.} \end{cases}$$

Note how submeasure (4) is composed from two submeasures of type (1). It is this kind of construction which also underlies the general case, treated in the subsequent Theorem 2.5. Informally, we then might describe the construction as the "ordinal sum" of submeasures of type (1) and (2).

In the following, a set $\{c_i\colon\ i \in I\}$ of pairwise disjoint non-zero elements of a boolean algebra such that $\bigvee_i c_i = 1$, is called a partition. Furthermore, for an element a of a boolean algebra $\mathcal{P}$, we denote the subalgebra of elements below a by $\mathcal{P}(a)$.

Theorem 2.5 *Let* $(\mathcal{P}; \leq, \neg, 0, 1)$ *be an atomistic, separable,* σ*-complete boolean algebra. Let* $(I; \leq)$ *be a countable linearly ordered set, and let* $\{c_i \colon\ i \in I\}$ *be a partition of* $\mathcal{P}$. *Associate to every* $i \in I$ *an interval* $[v_i, w_i] \subseteq [0,1]$ *such that* (α) $v_i < w_i$, (β) *for any* $j > i$ $\ w_i \leq v_j$, *and* (γ) $\bigvee_i w_i = 1$.

For every $i \in I$, *let* $\sigma_i \colon\ \mathcal{P}(c_i) \to [0,1]$ *be a standard submeasure. Let* $t = \sigma_i(e)$ *for an atom* e *below* c_i*; and let* $\tau_i \colon\ [t,1] \to [0,1]$ *be a linear function with positive coefficients such that* $v_i < \tau_i(t) < \tau_i(1) = w_i$. *Define* $\nu_i \colon\ \mathcal{P}(c_i) \to [0,1]$ *by* $\nu_i(0) = 0$ *and* $\nu_i(a) = \tau_i(\sigma_i(a))$ *for* $a > 0$.

Define now $\nu \colon\ \mathcal{P} \to [0,1]$ *as follows. Let* $\nu(0) = 0$. *For a non-zero* $a \in \mathcal{P}$, *let* $I_a = \{i \in I \colon\ a \cap c_i > 0\}$. *If* I_a *has the maximal element* j, *set* $\nu(a) = \nu_i(a \cap c_j)$*; otherwise set* $\nu(a) = \bigvee_{i \in I_a} v_i$. *Then* ν *is a normal submeasure on* $\mathcal{P}$ *which is continuous from below, homogeneous, and faithful.*

All normal submeasures on $\mathcal{P}$ *which are continuous from below, homogeneous, and faithful, arise in this way.*

Proof. The proof of the first part is somewhat tedious, but does not contain any difficult steps, so we skip it.

Assume that ν is a normal submeasure on $\mathcal{P}$ such that ν is continuous from below, homogeneous, and faithful. Let $I = \{\nu(e) \colon\ e \in \mathcal{A}(1)\}$; endow I with the natural order; and for each $i \in I$, let $c_i = \bigvee\{e \in \mathcal{A}(1) \colon\ \nu(e) = i\}$, so that $\{c_i \colon\ i \in I\}$ is a partition. Furthermore, let $v_i = \nu(\bigvee\{e \in \mathcal{A}(1) \colon\ \nu(e) < i\})$ and $w_i = \nu(c_i)$ for any $i \in I$. Note that I does not contain 0, because ν is strictly positive.

Let now $i \in I$, and assume that i is not minimal in I (note that I need not be have a minimal element at all). Choose an atom f such that $\nu(f) = i$, and let $e_1, e_2, \ldots$ be atoms such that $\nu(e_1), \nu(e_2), \ldots < \nu(f)$. We claim that then $\nu(\bigvee_k e_k) < \nu(f)$. Indeed, otherwise there are by homogeneity $\bar{e}_1, \bar{e}_2, \ldots \in \mathcal{P}$ such that $f \leq \bigvee_k \bar{e}_k$ and $\nu(\bar{e}_1), \nu(\bar{e}_2), \ldots < \nu(f)$. Since f is an atom, we would have $f \leq \bar{e}_l$ for some l, a contradiction.

It follows in particular $v_i < i$. This strict inequality holds evidently also for the case that i is minimal in I.

Next, let $e_1, e_2 \in \mathcal{A}(1)$ be distinct and of the same measure i. Then

$\nu(e_1 \vee e_2) > \nu(e_1)$; otherwise $e_1 \vee e_2$ as well as all non-zero elements strictly below $e_1 \vee e_2$ are of measure i, a contradiction to the faithfulness of ν.

Moreover, for some $i \in I$, let $f_1, \ldots, f_m$ be pairwise distinct atoms of measure i, let also $g_1, \ldots, g_n \in \mathcal{A}(1)$ be of measure i, and let $d_1, \ldots, e_1, \ldots \in \mathcal{A}(1)$ be of measure strictly smaller than i. We claim that then $\nu(f_1 \vee \ldots \vee f_m \vee \bigvee_k d_k) \leq \nu(g_1 \vee \ldots \vee g_n \vee \bigvee_k e_k)$ implies $m \leq n$. Indeed, by homogeneity $f_1 \vee \ldots \vee f_m \leq \bar{g}_1 \vee \ldots \vee \bar{g}_n \vee \bigvee_k \bar{e}_k$, where $\nu(\bar{g}_1) = \ldots = \nu(\bar{g}_n) = i$ and $\nu(\bar{e}_k) < i$ for all k. For $l = 1, \ldots, n$, due to the last paragraph, there cannot be two distinct atoms of measure i below $\bar{g}_l$. It follows that $f_1 \vee \ldots \vee f_m$ is below the supremum of at most n atoms of measure i and further atoms of measure strictly smaller than i. Because $f_1, \ldots, f_m$ are m distinct atoms, we conclude $m \leq n$.

Again, let $i \in I$, let $f_1, \ldots, f_n$ be pairwise distinct atoms of measure i, and let $e_1, \ldots$ be all the atoms of measure strictly smaller than i. We shall show that $\nu(f_1 \vee \ldots \vee f_n \vee \bigvee_k e_k) = \nu(f_1 \vee \ldots \vee f_n)$. From the last paragraph we know that $\nu(f_1 \vee \ldots \vee f_{n-1} \vee \bigvee_k e_k) < \nu(f_1 \vee \ldots \vee f_n)$. By Lemma 2.2(ii), there is a $g \in \mathcal{P}$ such that $\nu(f_1 \vee \ldots \vee f_{n-1} \vee g \vee \bigvee_k e_k) = \nu(f_1 \vee \ldots \vee f_n)$, and g can neither cover only atoms of measure strictly smaller than i, nor can g cover an atom of measure strictly larger than i; so g covers an atom of measure i, which is moreover distinct from $f_1, \ldots, f_{n-1}$. Now, if $\nu(f_1 \vee \ldots \vee f_n) < \nu(f_1 \vee \ldots \vee f_n \vee \bigvee_k e_k)$, there would be again some h which covers an atom of measure i distinct from $f_1, \ldots, f_{n-1}, g$, such that $\nu(f_1 \vee \ldots \vee f_{n-1} \vee g \vee h \vee \bigvee_k e_k) = \nu(f_1 \vee \ldots \vee f_n \vee \bigvee_k e_k)$, in contradiction to the last paragraph.

It follows in particular that $w_i = \nu(c_i) = \nu(\bigvee\{e \in \mathcal{A}(1) \colon\ \nu(e) \leq i\}) \leq v_j$ for any $j > i$. It moreover follows $\bigvee_i w_i = \nu(1) = 1$.

For any $i \in I$, let now ν_i be the restriction of ν to c_i. Clearly then, $\frac{1}{w_i}\nu_i$ is a normal submeasure; and by what we have shown, $\frac{1}{w_i}\nu_i$ is actually a standard submeasure. In particular, ν_i is a standard submeasure multiplied by a positive constant.

Let $a \in \mathcal{P}$ be such that $a \wedge c_i > 0$, but $a \wedge c_j = 0$ for every $j > i$, $i \in I$. We have shown that $\nu(a) = \nu(a \cap c_i)$, whence $\nu(a) = \nu_i(a \cap c_i)$. Moreover, let $a \in \mathcal{P}$ be such that there is no such i. From $a = \bigvee\{e \in \mathcal{A}(1) \colon\ e \leq a\}$, we conclude by continuity from below that $a = \bigvee\{v_i \colon a \cap c_i > 0\}$. $\square$

3 Similarity relations with respect to t-norms

We have dealt so far with a boolean algebra endowed with a metric, which is meant to express to what degree two elements of the algebra differ. We will now change our point of view in a straightforward way: As the metric d was also assumed to be bounded from above by 1, we may reverse the order of its range, that is, we may pass from d to a new binary function $E(\cdot,\cdot) = 1 - d(\cdot,\cdot)$. E then expresses the degree to which two elements resemble: $E(a,b)$ is the closer to the value 1 the more similar the two elements a and b are. So E is a binary fuzzy relation expressing similarity, that is: E is a similarity relation.

For an overview over the theory of similarity relations, we refer to [KlMePa]. Various different names have been used for this concept. We follow, in a slightly modified way, Zadeh's proposal; in [Hoe], "separated $[0,1]$-valued equalities" are studied; in [DeMe2], the notion "T-equality" is introduced, which is also used in [KlMePa]. For the connection between metrics and similarity relations in general, we refer to [Hoe]. The interplay between metrics and similarity relations w.r.t. t-norms which, like the archimedean continuous ones, have an additive generator, is studied in [DeMe1] and further elaborated in [DeMe3].

Definition 3.1 Let $\mathcal{P}$ be a boolean algebra.

(i) Let $\odot$ be a t-norm. Then we call $E\colon\ \mathcal{P} \times \mathcal{P} \to [0,1]$ a *similarity relation w.r.t.* $\odot$ if (α) $E(a,b) = 1$ iff $a = b$, (β) $E(a,b) = E(b,a)$, and (γ) $E(a,b) \odot E(b,c) \leq E(a,c)$ for $a,b,c \in \mathcal{P}$.

(ii) Let ν be a normal submeasure on $\mathcal{P}$ and let d_ν be the metric associated to ν. Then we call

$$E_\nu\colon\ \mathcal{P} \times \mathcal{P} \to [0,1], \quad (a,b) \mapsto 1 - d_\nu(a,b)$$

the *similarity relation on* $\mathcal{P}$ *induced by* ν.

The terminology of part (ii) is justified as follows [Hoe].

Proposition 3.2 *Let $\mathcal{P}$ be a boolean algebra, and let ν a normal submeasure on $\mathcal{P}$. Then E_ν is a similarity relation w.r.t. the Łukasiewicz t-norm.*

Moreover, let $\mathcal{P}$ be atomistic, and assume that $\nu(a) = \bigvee\{\nu(e) \colon e \in \mathcal{A}(a)\}$ for $a \in \mathcal{P}$. Then E_ν a similarity relation w.r.t. the minimum t-norm.

Proof. The first part is evident. For the second part, note that d_ν is an ultrametric. □

We now arrive at the main point of this paper: the question how the metric on a boolean algebra may give rise to a specific continuous t-norm, which is so-to-say characteristic for this metric. The idea is simple: Given a metric d, then $E = 1 - d$ is a similarity relation w.r.t. a t-norm $\odot$ if and only if

$$E(a, b) \odot E(b, c) \leq E(a, c) \tag{1}$$

holds for any $a, b, c \in \mathcal{P}$; so there might be a weakest t-norm fulfilling (1). This t-norm would have the property that the estimation of $E(a, c)$ out of $E(a, b)$ and $E(b, c)$ is the best possible.

If $E = E_\nu$ is based on a submeasure ν of the kind considered in Section 2, there is indeed a continuous t-norm which could be called the weakest t-norm fulfilling (1) up to isomorphism. However, this t-norm is not the natural choice, and we will proceed differently.

The operation $\odot$, as it appears in (1), is in general not used in total; actually only its restriction to the range of the similarity relation E is of interest. This gives reason to consider first an analogue of t-norms defined on a subset U of $[0, 1]$ only; we will call these operations U-based t-norms. What we have in mind is to determine the weakest ran(E)-based t-norm fulfilling (1). However, this will be only an intermediate step; the ran(E)-based t-norm will be extendable to an ordinary t-norm in a natural way.

Definition 3.3 Let U be a countable subset of the real unit interval containing 0 and 1. Then a binary operation $\odot_U$ on U is called a *U-based t-norm* if (α) $(U; \leq, \odot_U, 1)$ is an ordered commutative monoid, where $\leq$ is the natural order, and (β) for any pair a, b there is a largest element c such that $b \odot_U c \leq a$. Moreover, $\odot_U$ is called *continuous* if for any a, b such that $a \leq b$ there is an element c such that $b \odot_U c = a$.

In other words, a continuous U-based t-norm is the monoidal operation of a BL-algebra whose base set is the countable linearly ordered set U. Note furthermore that U-based t-norms, where U is finite, are discrete t-norms, which were studied by several authors, see [KlMePa].

In what follows, we will freely apply Definition 3.1 also to the case that the involved t-norm $\odot$ is U-based, where $U = \text{ran}(E)$.

Furthermore, a t-norm $\odot_1$ - an ordinary or a U-based one - will be called weaker than a second one $\odot_2$ if $a \odot_1 b \geq a \odot_2 b$ for all a, b.

Definition 3.4 Let $\mathcal{P}$ be a boolean algebra, and let E_ν: $\mathcal{P} \times \mathcal{P} \to [0, 1]$ be the similarity relation induced by some submeasure ν on $\mathcal{P}$. Let $U = \text{ran}(E_\nu)$, and let $\odot_U$ be a continuous U-based t-norm. We say that E_ν *determines* $\odot_U$ if $\odot_U$ is the weakest among all continuous U-based t-norms w.r.t. which E_ν is a similarity relation.

As a first step, we formulate the problem how a similarity relation induced by a submeasure determines a t-norm, directly with respect to the involved submeasure.

Lemma 3.5 *Let ν be a normal submeasure on a complete boolean algebra $\mathcal{P}$. Then E_ν is a similarity relation w.r.t. the t-norm $\odot$ if and only if*

$$\nu(a) \oplus \nu(b) \geq \nu(a \vee b) \text{ for } a, b \in \mathcal{P} \text{ such that } a \wedge b = 0, \qquad (2)$$

where $\oplus$ is the t-conorm dual to $\odot$.

This is why we will work in the sequel with t-conorms instead of t-norms. We will assume that all definitions given for t-norms are dually also defined for t-conorms. This applies moreover also to U-based t-norms, which correspond to V-based t-conorms, where $V = 1 - U = \{1 - u\colon\ u \in U\}$.

Now, the first step for the solution of our problem is as follows.

Theorem 3.6 *Let $\mathcal{P}$ be an atomistic, separable, σ-complete boolean algebra. Let ν be a normal submeasure on $\mathcal{P}$ which is continuous from below, homogeneous, and faithful. Let $V = \operatorname{ran}(\nu)$. Then we may define*

$$v \oplus_V w = \max \{\nu(a \vee b)\colon\ \nu(a) = v,\ \nu(b) = w\} \tag{3}$$

for $v, w \in V$. Moreover, let $U = \operatorname{ran}(E_\nu) = 1 - V$ and $v \odot_U w = 1 - ((1 - v) \oplus_V (1 - w))$ for $v, w \in U$; then $\odot_U$ is the continuous U-based t-norm determined by E_ν.

Proof. According to the analysis of ν in Theorem 2.5, $\oplus_V$ may be described as follows. There are a linearly ordered set $(I; \leq)$ and pairs $v_i, w_i \in V$ for every $i \in I$ such that $v_i < w_i$ and $w_i \leq v_j$ if $i < j$. Moreover, for every i, let $V_i = [v_i, w_i] \cap V = \{x_i^{(k)}\colon\ k \leq \lambda_i\}$, where $1 \leq \lambda_i \leq \omega$ and $v_i = x_i^{(0)} < x_i^{(1)} < x_i^{(2)} < \ldots < x_i^{(\lambda_i)} = w_i$. Then $\oplus_V$ acts on V_i as follows: $x_i^{(k)} \oplus_V x_i^{(l)} = x_i^{(k+l)}$ in case $k, l < \omega$ and $k + l \leq \lambda_i$; else the result is $x_i^{(\lambda_i)}$. Finally, $\oplus_V$ puts any two elements of V not lying both in an interval $[v_i, w_i]$ for some i, to their supremum. So it follows that $\oplus_V$ is a V-based t-conorm, and it is in particular evident that the maximum in (3) always exists.

It is furthermore clear that $\oplus_V$ is among all functions fulfilling (2) the minimal one. So the remaining part of the assertion follows. □

Let $\odot_U$ be the U-based t-norm associated to a submeasure according to Theorem 3.6. The analysis of the preceeding proof shows that $\odot_U$ is the ordinal sum of three kinds of operations: the boolean AND on $\{0, 1\}$; the Łukasiewicz conjunction on $\{0, \frac{1}{n}, \ldots, 1\}$ for some $n \geq 2$; and the product conjunction on $\{(\frac{1}{2})^n\colon\ n < \omega\} \cup \{0\}$. So there is natural way how to extend $\odot_U$ to a continuous t-norm; we may simply use the minimum t-norm in the first case, the Łukasiewicz t-norm in the second case, and the product t-norm in the third case.

However, there are more possibilities, and we will give an axiomatic characterisation of the desired case.

For clarity, we will mark intervals in $U \subseteq [0, 1]$ by a suffix, that is, $[a, b]_U \stackrel{\text{def}}{=} \{x \in U\colon\ a \leq x \leq b\}$ for $a, b \in U$.

Definition 3.7 For some $U \subseteq [0,1]$, let $\odot_U$ be a continuous U-based t-norm.

(i) Let $a, b, c, d \in U$ such that $a < b$ and $c < d$. We will say that the intervals $[a,b]_U, [c,d]_U \subseteq U$ are *injective* for $\odot_U$ if for all $e \in [a,b]_U$ and all $f \in [c,d]_U$, the mappings $[c,d]_U \to U,\ \ x \mapsto e \odot_U x$ and $[a,b]_U \to U,\ \ x \mapsto f \odot_U x$ are injective.

(ii) Let $\odot$ be a continuous t-norm extending $\odot_U$. Then we say that $\odot$ extends $\odot_U$ *smoothly* if, for all $a, b \in [0,1]$, we have $a \odot b < a, b$ exactly in the case that there are intervals $[a_0, a_1]_U, [b_0, b_1]_U \subseteq U$ injective for $\odot_U$, such that $a < a_1$, $b < b_1$, and $a \odot b \geq a_0 \odot b_0$.

It will turn out that for smoothness implies uniqueness of the extension, and we will base the main definition on this notion.

Definition 3.8 Let $\mathcal{P}$ be a boolean algebra, and let $E_\nu\colon\ \mathcal{P} \times \mathcal{P} \to [0,1]$ be the similarity relation induced by some submeasure ν on $\mathcal{P}$. Let $U = \operatorname{ran}(E_\nu)$, and let $\odot_U$ be the continuous U-based t-norm determined by E_ν. We say that E_ν *determines* the continuous t-norm $\odot$ if $\odot$ smoothly extends $\odot_U$.

Continuous t-conorms are ordinal sums of the three standard t-conorms; we will call the latter Łukasiewicz conorm, product conorm, and maximum conorm, respectively.

Theorem 3.9 *Let $(\mathcal{P}; \leq, \neg, 0, 1)$ be an atomistic, separable, σ-complete boolean algebra. Let ν be a normal submeasure on $\mathcal{P}$ which is continuous from below, homogeneous, and faithful. Then E_ν determines a continuous t-norm, which is unique up to isomorphism.*

Moreover, up to isomorphism, all continuous t-norms arise in this way.

Proof. Let $V = 1 - \operatorname{ran}(E_\nu)$, and let $\oplus_V$ be the V-based t-conorm defined by (3). We shall use the same notation as in the proof of Theorem 3.6; in particular, $(I; \leq)$ is a linear order and, for every $i \in I$, $u_i, v_i \in V$. It is obvious how we may $\oplus_V$ extend to a t-conorm $\oplus$. Namely, we extend, for any $i \in I$, the restriction of $\oplus_V$ to V_i to the whole interval $[v_i, w_i]$ as follows: in case $\lambda_i = 1$, we let $\oplus$ on $[v_i, w_i]$ be the maximum conorm; in case $2 \leq \lambda_i < \omega$, we take (an isomorphic copy of) the Łukasiewicz conorm, and in case $\lambda_i = \omega$, we take (an isomorphic copy of) the product conorm. In all other cases, we set $\oplus$ to the maximum of the two arguments.

It is not difficult to see that $\oplus$ extends $\oplus_V$ smoothly; so E_ν determines $\oplus$. We claim that $\oplus$ is up to isomorphism the only continuous t-norm with this property. Indeed, let $\oplus'$ be any smooth extension of $\oplus_V$ to a continuous t-conorm. Then $\oplus'$ has, like $\oplus$, the idempotents v_i and w_i for all $i \in I$. Consider first an interval $[v_i, w_i]$ such that $\lambda_i \geq 2$. Then $\oplus'$ clearly cannot have any idempotent within $[x_i^{(1)}, w_i)$. Furthermore, the intervals $[v_i, x_i^{(1)}]_V, [v_i, x_i^{(1)}]_V \subseteq V$ are injective for $\oplus_V$; it follows that $\oplus'$ cannot have an idempotent in $(v_i, x_i^{(1)})$ either. So $\oplus'$ is on $[v_i, w_i]$ isomorphic to $\oplus$. Consider second two idempotents v, w such that $v < w$ and such that (v, w) is disjoint from all $[v_i, w_i]$ such that $\lambda_i \geq 2$. Then $a \odot b > a, b$ for any $a, b \in [v, w]$ contradicts smoothness, because injective intervals are necessarily within some $[v_i, w_i]_U$, where $\lambda_i \geq 2$. It follows that $\oplus'$ takes, like $\oplus$, the maximum values outside the intervals $[v_i, w_i]$ such that $\lambda_i \geq 2$.

The fact that all continuous t-norms arise in this way, is an immediate consequence of the representation theorem of continuous t-conorms by means of ordinal sums. $\square$

4 Conclusion

We found a way to motivate the usage of any specific continuous t-norm. Our framework are certain metric boolean algebras; such algebras are naturally endowed with a similarity relation w.r.t. several continuous t-norms; we have shown how to choose the weakest one among them and that this can be every continuous t-norms.

To conclude the paper, let us have a look on related and possible further research. The framework which we have chosen is of a rather specific nature, and it would be clearly desirable to explore further possibilities. It actually would be desirable to define a general abstract framework within which fuzzy-logical connectives would find their natural interpretation – comparably to the case of modal logics and Kripke models. Few efforts in this direction were made. For instance, in [Par], a formalism is developed in which alternatively the usage of the minimum t-norm or the product t-norm naturally appears. Searching for a motivation to use continuous t-norms, we may certainly also go the opposite way: we may wonder which t-norms have which kind of practical use. Systematic work in this direction can be found in [BaNa].

In view of the present work itself, we note that several questions remained open. We are in particular interested in the question if there is an analogous version of Theorem 3.9 which assumes the boolean algebra just to be separable and σ-complete. This would result in a much more general framework. Unfortunately, the present results are not of much use for this aim, being heavily dependent on the atomicity condition.

Furthermore, it would clearly be desirable to characterise a wider class of t-norms, like e.g. the left-continuous ones, in a similar way as proposed here. However, this will be difficult as long as we are lacking a general structure theory.

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